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Some statistical properties of surface slopes via remote sensing considering a non-Gaussian probability density function

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\textbf{ABSTRACT}

Theoretical relationships of statistical properties of surface slope from statistical properties of the image intensity in remotely sensed images, considering a non-Gaussian probability density function of the surface slope, are shown. Considering a variable detector line of sight angle and considering ocean waves moving along a single direction and that the observer and the sun are both in the vertical plane containing this direction, new expressions, using two different glitter functions, between the variance of the intensity of the image and the variance of the surface slopes are derived. In this case, skewness and kurtosis moments are taken into account. However, new expressions between correlation functions of the intensities in the image and surface slopes are numerically analyzed; for this case, the skewness moments were considered only. It is possible to observe more changes in these statistical relationships when the Rect function is used. The skewness and kurtosis values are in direct relation with the wind velocity on the sea surface.

\textbf{KEYWORDS}

Glitter pattern; aerial camera; images; sea surface; statistical moments; probability density functions

\textbf{ARTICLE HISTORY}

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\textbf{1. Introduction}

Barber [1] showed that the directionality of ocean waves can be estimated from the diffraction patterns of photographic image of the sea surface. Cox and Munk [2,3] studied the distribution of intensity in aerial photographs of the sea. One of their conclusions was that for wind speed about 10 m/s, the probability density function of the surface slope contains information about the skewness and kurtosis moments. Longuet-Higgins [4] studied the nonlinearity of ocean waves considering non-Gaussian probability functions for the distribution of sea surface slopes.

The non-Gaussian sea surface slopes model is very important because it includes information about the skewness and kurtosis in the statistics of waves. These moments appear for wind speed above 10 m/s on the sea surface. The waves tend to have positive skewness [4], and sometimes negative skewness [3].

In a first approximation, the sea surface slopes have a Gaussian probability density function; in a second approximation, skewness is taken into account, the kurtosis is zero, as are all the higher moments, and in a third approximation the kurtosis is taken into account in the distribution [4].

In this article, we assume that all the ocean waves are moving along a single direction, and the vertical plane containing this direction includes the observer and the sun (1D-case). In this theoretical work, a sensor was placed at different heights: 100, 500, 1000, and 5000 m. The sensor received reflected light coming from the surface slopes for different incidence angles from 10° to 50°.

In this work were used a Rect and a Gaussian glitter functions [5–7]. New expressions, using two different glitter functions, between the variance of the intensity of the image and the surface slopes for different incidence angles from 10° to 50°.

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2. Statistical estimation on images of glitter patterns

2.1. Optical model

In order to obtain the statistical relationships between the intensities of the light on the sea surfaces vs. the sea surface slopes, the same model presented by Álvarez-Borrego and Martín-Atienza [6] was used.

In Figure 1, \((\theta_d)\), is a variable angle subtended by the optical system of the detector with the normal to a point of the surface. In addition, this model does not impose
a restriction on the sensor field of view, and the effect neglected of shadowing, i.e., when some reflecting points are obscured by roughness of the sea surface. This angle is given by

$$\theta_i = \tan^{-1}\left(\frac{i \Delta x}{H}\right)$$  \hspace{1cm} (1)

where \(\theta_i\), is the angle subtended by the image sensor with the normal to point \(i\) on the surface and \(\theta_s\) is the angle of incidence light on sea respect to normal to sea surface, \(H\) is the height variable of the detector, \(\Delta x\) is the interval between surface points, and \(\theta_o\) is the angle subtended between the normal to the mean surface and the normal to the slope for each \(i\) point in the surface, i.e.,

$$\alpha_i = \frac{\theta_i}{2} + \frac{1}{2} \tan^{-1}\left(\frac{i \Delta x}{H}\right)$$  \hspace{1cm} (2)

In this more realistic physical situation, the angle \(\theta_i\) is changing with respect to each point in the surface.

The relationship between the statistics of the intensity in the image and surface slopes is derived in this case, considering a non-Gaussian probability density function for slopes, given by Álvarez-Borrego and Martín-Atienza [8]

$$p(M_i) = \frac{1}{\sigma_M \sqrt{2\pi}} \exp\left(-\frac{(M_i - M_{oi})^2}{2\sigma_M^2}\right) \left[1 + \frac{1}{6} \lambda_i^{(3)} + \frac{1}{24} \lambda_i^{(4)} \left(\frac{M_i - M_{oi}}{\sigma_M}\right)^2 \right]$$  \hspace{1cm} (3)

where \(\lambda_i^{(3)}\) and \(\lambda_i^{(4)}\) are skewness and kurtosis, respectively, \(M_i = \tan(\alpha_i)\) and \(\sigma_M\) is the standard deviation of the slope values, \(M_i\) is the slope value in the point \(i\).

According with Álvarez-Borrego [5], Álvarez-Borrego and Martín-Atienza [6] and Poom-Medina et al. [7], the Rect and the Gaussian glitter functions can be expressed as

$$B_R(M_i) = \text{Rect}\left[\frac{M_i - M_{oi}}{\frac{1}{2}(1 + M_{oi})} \beta_1/2\right]$$  \hspace{1cm} (4)

$$B_G(M_i) = \exp\left[-\frac{(M_i - M_{oi})^2}{\sigma_1^2}\right]$$  \hspace{1cm} (5)

where \(\beta_1 = \frac{1}{2}(1 + M_{oi})\) \(\beta_1/2\) gives information of the width of the Gaussian glitter function and \(\beta_1\) is the apparent diameter of the source and \(M_{oi} = \tan\left[\frac{\theta_o + \theta_s}{2}\right]\) is a definition only. However, the interval characterized by (4) and (5) define a specular band where certain slopes generate spots in the image given by

$$\mu_i = \frac{1}{N} \sum_{i=1}^{N} \text{Rect}\left[\frac{M_i - M_{oi}}{\frac{1}{2}(1 + M_{oi})} \beta_1/2\right] \exp\left(-\frac{M_i^2}{2\sigma_1^2}\right) \left[1 + \frac{1}{6} \lambda_i^{(3)} + \frac{1}{24} \lambda_i^{(4)} \left(\frac{M_i - M_{oi}}{\sigma_1}\right)^2 \right] dM_i$$  \hspace{1cm} (6)

in accordance with [9], that is the slope interval, where a bright spot is received by the image sensor.

2.2. The variance case: relationship between the variance of the intensities in the image and surface slopes considering a non-Gaussian probability density function for the slopes

In order to obtain the image variance corresponding to \(i\) position, the mean of the image \(\mu_i\) can be written as \(8\)

$$\mu_i = \langle I(x) \rangle = \frac{1}{N} \sum_{i=1}^{N} B(M_i) p(M_i) dM_i$$  \hspace{1cm} (7)

where if first, we consider, \(B_R(M_i)\) like the Rect glitter function defined by Equation (4) and \(p(M_i)\) is the non-Gaussian probability density function in one dimension, substituting Equations (3) and (4) in Equation (8) gives

$$\mu_i = \frac{1}{N} \sum_{i=1}^{N} \left[ \text{Rect}\left[\frac{M_i - M_{oi}}{\frac{1}{2}(1 + M_{oi})} \beta_1/2\right] \right] \exp\left(-\frac{M_i^2}{2\sigma_1^2}\right) \times \left[1 + \frac{1}{6} \lambda_i^{(3)} + \frac{1}{24} \lambda_i^{(4)} \left(\frac{M_i - M_{oi}}{\sigma_1}\right)^2 \right] dM_i$$  \hspace{1cm} (8)
where \( L_1 \) and \( L_2 \) are defined like \( L_1 = M_{lo} - (1 + M_{lo}^2) \beta/4 \) and \( L_2 = M_{lo} + (1 + M_{lo}^2) \beta/4 \), respectively, and if it is defined \( A = \frac{1}{2\sigma_M} \), the Equation (9) can be written as

\[
\mu_i = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\sigma_M \sqrt{2\pi}} \int_{L_i} \exp \left[-\left(\frac{1}{8} \lambda_M^{(i)} \right) \right] \left[ \left( 1 + \frac{1}{8} \lambda_M^{(i)} \right) - \frac{L_i}{2\sigma_M} - \frac{L_i \lambda_M^{(i)}}{4\sigma_M^3} + \frac{L_i \lambda_M^{(i)}}{6\sigma_M^5} + \frac{L_i \lambda_M^{(i)}}{24\sigma_M^7} \right] \, dM_i
\]

and considering the definitions

\[
P = \frac{\left( 1 + \frac{1}{8} \lambda_M^{(i)} \right) \sigma_M \sqrt{2\pi}}{2\sigma_M^2 \sqrt{2\pi}}; \quad Q = -\frac{\lambda_M^{(i)}}{2\sigma_M^2 \sqrt{2\pi}}; \quad R = -\frac{\lambda_M^{(i)}}{4\sigma_M^3 \sqrt{2\pi}}; \quad S = \frac{\lambda_M^{(i)}}{6\sigma_M^4 \sqrt{2\pi}}; \quad T = \frac{\lambda_M^{(i)}}{24\sigma_M^5 \sqrt{2\pi}};
\]

the result is

\[
\mu_i = \sum_{i=1}^{N} \frac{1}{8NA^{7/2}} \exp \left[-\left(\frac{1}{8} \lambda_M^{(i)} \right) \right] \left( \sqrt{\pi} \exp \left[\frac{AL_i}{2\sigma_M} \right] \right) \left[ \left( \sqrt{AL_i} \right) \exp \left[\frac{AL_i}{2\sigma_M} \right] \right] \left( 4A^2P + 2AR + 3T \right) - 2A^2 \left( \frac{2A^2(2A^2 + 3T)}{24\sigma_M^7} \right)
\]

and the variance of the intensities in the image is defined by

\[
\sigma_i^2 = \frac{1}{N} \sum_{i=1}^{\infty} [B_i(M_i) - \mu_i]^2 P(M_i) \, dM_i
\]

and considering the definitions similar to Equation (11), the result is

\[
\sigma_i^2 = \mu_i (1 - \mu_i).
\]

Equation (13) is the required relation between the variance of the intensities in the image, \( \sigma_i^2 \), and the variance of the surface slopes, \( \sigma_M^2 \), when a Rect glitter function is considered.

Now, if we consider the Gaussian glitter function, Equation (5)

\[
\mu_i = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\sigma_M \sqrt{2\pi}} \int_{L_i} \exp \left[-\left(\frac{1}{8} \lambda_M^{(i)} \right) \right] \left[ \left( 1 + \frac{1}{8} \lambda_M^{(i)} \right) - \frac{L_i}{2\sigma_M} - \frac{L_i \lambda_M^{(i)}}{4\sigma_M^3} + \frac{L_i \lambda_M^{(i)}}{6\sigma_M^5} + \frac{L_i \lambda_M^{(i)}}{24\sigma_M^7} \right] \, dM_i
\]

and considering the definitions

\[
P = \frac{\left( 1 + \frac{1}{8} \lambda_M^{(i)} \right) \sigma_M \sqrt{2\pi}}{2\sigma_M^2 \sqrt{2\pi}}; \quad Q = -\frac{\lambda_M^{(i)}}{2\sigma_M^2 \sqrt{2\pi}}; \quad R = -\frac{\lambda_M^{(i)}}{4\sigma_M^3 \sqrt{2\pi}}; \quad S = \frac{\lambda_M^{(i)}}{6\sigma_M^4 \sqrt{2\pi}}; \quad T = \frac{\lambda_M^{(i)}}{24\sigma_M^5 \sqrt{2\pi}};
\]

the result is

\[
\mu_i = \sum_{i=1}^{N} \frac{1}{8NA^{7/2}} \exp \left[-\left(\frac{1}{8} \lambda_M^{(i)} \right) \right] \left( \sqrt{\pi} \exp \left[\frac{AL_i}{2\sigma_M} \right] \right) \left[ \left( \sqrt{AL_i} \right) \exp \left[\frac{AL_i}{2\sigma_M} \right] \right] \left( 4A^2P + 2AR + 3T \right) - 2A^2 \left( \frac{2A^2(2A^2 + 3T)}{24\sigma_M^7} \right)
\]

and the variance of the intensities in the image is defined by

\[
\sigma_i^2 = \frac{1}{N} \sum_{i=1}^{\infty} [B_i(M_i) - \mu_i]^2 P(M_i) \, dM_i
\]

and considering the definitions similar to Equation (11), the result is

\[
\sigma_i^2 = \mu_i (1 - \mu_i).
\]

[8] by
The product of Rect glitter function is

\[ M_{oj} = \tan \left( \frac{\theta_{ij} + \theta_{ij}}{2} \right) \]

and

\[ \sigma_{2} = \sum_{i=1}^{N} \int_{-\infty}^{\infty} \left[ B_G(M_i) - \mu_i \right]^2 \frac{p(M_i)}{\sigma_i^2} dM_i \]

\[ = \sum_{i=1}^{N} \int_{-\infty}^{\infty} \left[ B_G(M_i) \right] \frac{p(M_i)}{\sigma_i^2} dM_i - \mu_i^2. \tag{17} \]

The solution of the integral in Equation (17) is similar to Equation (16), but now with

\[ A = \frac{4\sigma_i^2 + \theta_i^2}{2\sigma_i^2 \theta_i^2}, \quad B = -\left( \frac{2M_0}{\theta_i^2} \right) \]

and

\[ C = \frac{2M_0}{\theta_i^2}, \quad \mu_i \text{ is like Equation (16). Equation (17) is the} \]

required relation between the variance of the intensities in the image, \( \sigma_I^2 \), and the variance of the surface slopes, \( \sigma_M^2 \), when a Gaussian glitter function is considered.

### 2.3. The correlation case: relationship between the correlation of the intensities in the image and surface slopes considering a non-Gaussian probability density function for the slopes

The relationship between the correlation function of the surface slopes, \( C_M(\tau) \), and the correlation function of the glitter pattern, \( C_I(\tau) \), is given by

\[ \sigma_I^2 C_I(\tau) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ B_R(M_{1j}) B_R(M_{2j}) p(M_{1j}, M_{2j}) dM_{1j} dM_{2j} \right. \]

\[ \left. - \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{-\infty}^{\infty} B(M_{1j}) B(M_{2j}) p(M_{1j}, M_{2j}) dM_{1j} dM_{2j} \right]. \tag{18} \]

The product of Rect glitter function is

\[ B_R(M_{1j}) B_R(M_{2j}) = \text{Rect} \left[ \frac{M_{1j} - M_{2j}}{(1 + M_{1j}^2) \beta / 2} \right] \]

\[ \times \text{Rect} \left[ \frac{M_{2j} - M_{oj}}{(1 + M_{2j}^2) \beta / 2} \right], \tag{19} \]

where

\[ M_{oj} = \tan \left( \frac{\theta_{ij} + \theta_{ij}}{2} \right) \]

and

\[ \frac{\theta_{ij} + \theta_{ij}}{2} \]
Figure 3. Comparison between the variances of intensities of the image, $\sigma_i^2$, vs. the variance of the surface slopes, $\sigma_M^2$, considering: (a) Rect glitter function with skewness of $-0.463$ and kurtosis of $0.0$, (b) Rect glitter function with skewness of $-0.463$ and kurtosis of $0.215$, (c) Gaussian glitter function with skewness of $-0.463$ and kurtosis of $0.0$, (d) Gaussian glitter function with skewness of $-0.463$ and kurtosis of $0.215$. The detector is in $H = 500$ m.

$$p(M_1, M_2) = \frac{1}{2\sigma_M^2 \sqrt{1-C_i^2(r)}} \exp \left[ -\frac{M_1^2 + M_2^2 - 2C_i(r)M_1M_2}{2\sigma_M^2 [1-C_i^2(r)]} \right] \times$$

$$\begin{align*}
&\begin{cases}
\lambda_{M_1}^{(30)} \left( \frac{M_1}{\sigma_M} \right)^3 - 3\sigma_M^2 \left( \frac{M_1}{\sigma_M} \right) + \\
3\lambda_{M_2}^{(21)} \left( \frac{M_2}{\sigma_M} \right)^2 \left( \frac{M_2}{\sigma_M} \right)^2 - \sigma_M^2 \left( \frac{M_2}{\sigma_M} \right) + 2\sigma_M^2 C_M \left( \frac{M_2}{\sigma_M} \right) + \\
3\lambda_{M_2}^{(12)} \left( \frac{M_2}{\sigma_M} \right) \left( \frac{M_2}{\sigma_M} \right)^2 - \sigma_M^2 \left( \frac{M_2}{\sigma_M} \right) + 2\sigma_M^2 C_M \left( \frac{M_2}{\sigma_M} \right) + \\
\lambda_{M_1}^{(03)} \left( \frac{M_1}{\sigma_M} \right)^3 - 3\sigma_M^2 \left( \frac{M_1}{\sigma_M} \right) + \\
3\lambda_{M_1}^{(21)} \left( \frac{M_1}{\sigma_M} \right)^2 \left( \frac{M_1}{\sigma_M} \right)^2 - \sigma_M^2 \left( \frac{M_1}{\sigma_M} \right) + 2\sigma_M^2 C_M \left( \frac{M_1}{\sigma_M} \right) + \\
\lambda_{M_1}^{(12)} \left( \frac{M_1}{\sigma_M} \right) \left( \frac{M_1}{\sigma_M} \right)^2 - \sigma_M^2 \left( \frac{M_1}{\sigma_M} \right) + 2\sigma_M^2 C_M \left( \frac{M_1}{\sigma_M} \right) + \\
\end{cases}
\end{align*}$$

If we define

$$A = \frac{1}{2\sigma_M^2 (1-C_i^2)}, \quad B = -\frac{C_i M_1}{2\sigma_M^2 (1-C_i^2)},$$

$$C = \frac{1}{2\sigma_M^2 (1-C_i^2)};$$

$$E = \frac{1}{2\sigma_M^2 \sqrt{1-C_i^2}}; \quad F = 1 + \frac{1}{6} M_1^3 - 3\sigma_M^2 M_1;$$

$$S = -\frac{1}{2} \lambda_{M_1}^{(30)} \sigma_M + \lambda_{M_1}^{(21)} \sigma_M C_M + \frac{1}{2} \lambda_{M_1}^{(12)} \frac{M_1^2}{\sigma_M^2} - \frac{1}{2} \lambda_{M_1}^{(12)} \sigma_M;$$

$$P = \frac{\lambda_{M_1}^{(03)}}{\sigma_M^2}; \quad Q = \frac{\lambda_{M_1}^{(21)}}{2\sigma_M^2}.$$
Rewriting the Equation (18), we have

\[ \sigma^2_{C_i}(\tau) = \frac{E}{N^2 \sigma_j^2} \sum_{j=1}^{N} \sum_{i=1}^{L_i} \exp \left\{ - \left( A M_i^2 + 2B M_i + C \right) \right\} \times \left\{ P M_i^3 + Q M_i^2 + S M_i + F \right\} dM_i dM_j, \]  

(22)

The result is

\[ \sigma^2_{C_i}(\tau) = \frac{E}{4N^2 \sigma_j^2} \sum_{j=1}^{N} \sum_{i=1}^{L_i} \exp \left\{ - \left( A L_j^2 + 2B L_j + C \right) \right\} \times \left\{ \sqrt{\pi} \exp \left[ \frac{(A L_j + B)^2}{A} \right] \text{erf} \left[ \frac{A L_j + B}{\sqrt{A}} \right] \times \right. \]

\[ \left. \begin{array}{c}
(2A^2F + A^2(Q - 2BS) + AB(2BQ - 3P) - 2B^3 P) \\
-2 \sqrt{A} \left( A^2 (L_j (PL_{j2} + Q) + S) + A ( -BPL_{j2} - BQ + P) + B^2 P \right) \\
\sqrt{\pi} \exp \left[ \frac{(A L_j + B)^2}{A} \right] \text{erf} \left[ \frac{A L_j + B}{\sqrt{A}} \right] \times \\
\left(2A^2F + A^2(Q - 2BS) + AB(2BQ - 3P) - 2B^3 P \right) \\
-2 \sqrt{A} \left( A^2 (L_j (PL_{j2} + Q) + S) + A ( -BPL_{j2} - BQ + P) + B^2 P \right) \end{array} \right\}, \]

(23)

This last equation is solved numerically. Equation (23) is the required relation between the correlation of the

intensities in the image and the correlation of the sea surfaces slopes when a Rect glitter function is used.

Now, the product of Gaussian glitter functions is

\[ B_G(M_{ij1}) B_G(M_{ij2}) = \exp \left[ \frac{(M_{ij1} - M_{ij2})^2}{\theta_i^2} \right] \times \exp \left[ \frac{(M_{ij2} - M_{ij3})^2}{\theta_j^2} \right], \]  

(24)

Figure 4. Comparison between the variances of intensities of the image, \( \sigma^2_{I} \), vs. the variance of the surface slopes, \( \sigma^2_{M} \), considering: (a) Rect glitter function with skewness of −0.463 and kurtosis of 0.0, (b) Rect glitter function with skewness of −0.463 and kurtosis of 0.215, (c) Gaussian glitter function with skewness of −0.463 and kurtosis of 0.0, (d) Rect glitter function with skewness of −0.463 and kurtosis of 0.215. The detector is in \( H = 1000 \) m.
where \( \theta_j = \left( 1 + M_{\theta_j}^2 \right) \theta_j \).

Now, taking the non-Gaussian probability density function [8], Equation (20), and if we define

\[
A = \frac{1}{\theta_j^2} + \frac{1}{2 \sigma_j^2 (1-C_j^2)}, \quad B = \left( \frac{M_j}{\theta_j^3} + \frac{C_j M_j}{2 \sigma_j^2 (1-C_j^2)} \right), \quad C = \frac{M_j^2}{\theta_j^4} + \frac{M_j^2}{\theta_j^4} - \frac{2 M_j M_{\theta_j}}{\theta_j^4} + \frac{2 \sigma_j M_{\theta_j}}{\sigma_j^2 (1-C_j^2)};
\]

\[
E = \frac{1}{2 \sigma_j^2 \sqrt{1-C_j^2}}; \quad F = 1 + \frac{M_j^3}{6 \sigma_j^2} \left( \frac{M_j}{\sigma_j^2} \right)^3 - 3 \sigma_j M_{\theta_j} \right) + \left( \frac{M_j}{\sigma_j^2} \right)^2 \frac{C_j}{C_j^2} \sigma_j M_{\theta_j}^2; \]

\[
S = -\frac{1}{2} \Lambda_M^{(3)} \sigma_j + \frac{1}{2} \Lambda_M^{(2)} \sigma_j \sigma_{M} + \frac{1}{2} \Lambda_M^{(1)} \frac{M_j^2}{\sigma_j^2} \sigma_{M}^2 - \frac{1}{2} \Lambda_M^{(1)} \sigma_j^2; \]

\[
P = \frac{\sigma_j^2 M_{\theta_j}^2}{2 \sigma_j^2}, \quad Q = \frac{\sigma_j^2 M_{\theta_j}^2}{2 \sigma_j^2},
\]

and by rewriting the Equation (18), we have

\[
\sigma_i^2 C_i(\tau) = \frac{E}{N^2} \sum_{j=1}^{N} \sum_{l=1}^{L_j} \sum_{i=1}^{L_i} \exp \left[ - (A M_{ij}^2 + 2 B M_{ij} + C) \right] \times \{ P M_{ij}^2 + Q M_{ij}^2 + S M_{ij} + F \} \, dM_{ij} \, dM_{ij},
\]

where the result is similar to Equation (23), but with the new definitions given by Equation (25). Of this way, we obtain the required relation between the correlations of the intensities of the image with the correlation function of the sea surface slopes when a Gaussian glitter function is used.

### 3. Results and discussions

The relationships between the variances of the images with respect to the variance of the sea surface slopes, using a non-Gaussian probability density function, where the skewness and the kurtosis were taken into account are shown in Figures 2–5. In these graphs is showed how the contributions due to skewness and the kurtosis have a slight displacement, with respect to the curves previously reported [6], this displacement is more obvious when the Rect glitter function is used. The values used in these calculations are \( \Lambda_M^{(3)} = -0.463, \Lambda_M^{(4)} = 0.215 \). Figures 2–5 show these relationships for different heights of the sensor.

**Figure 5.** Comparison between the variances of intensities of the image, \( \sigma_i^2 \), vs. the variance of the surface slopes, \( \sigma_{M}^2 \), considering: (a) Rect glitter function with skewness of \(-0.463\) and kurtosis of \(0.0\), (b) Rect glitter function with skewness of \(-0.463\) and kurtosis of \(0.215\), (c) Gaussian glitter function with skewness of \(-0.463\) and kurtosis of \(0.0\), and (d) Gaussian glitter function with skewness of \(-0.463\) and kurtosis of \(0.215\). The detector is in \( H = 5000 \) m.
The process was explained in Poom-Medina, et al. [7]. The behavior of the curves is the same for all the cases, Figure 3. In $H = 1000$ m, the inversion process is ended, Figure 4, and for $H = 5000$ m, the process looks very complete. In each case, the curves showing the same pattern do not matter if the glitter pattern was modeled with the Rect or the Gaussian functions. The Rect glitter function gives higher values for the variances of the intensities of the sea surface slopes can be calculated. When $H = 100$ m, for small values of $\sigma^2_M$, it is possible to find higher values for $\sigma^2_I$, Figure 2. Figure 2(c)–(d) shows almost the same results (the difference can be seen numerically only). In $H = 500$ m, the inverse process can be observed. This process was explained in Poom-Medina, et al. [7]. The behavior of the curves is the same for all the cases, Figure 3. In $H = 1000$ m, the inversion process is ended, Figure 4, and for $H = 5000$ m, the process looks very complete. In each case, the curves showing the same pattern do not matter if the glitter pattern was modeled with the Rect or the Gaussian functions. The Rect glitter function gives higher values for the variances of the intensities of the

Figure 6. Comparison between the correlations of intensities of the image vs. the correlations of the surfaces slopes considering Rect glitter function (left side) and Gaussian glitter function (right side). $H = 100$ m (a and b), $H = 500$ m (c and d), $H = 1000$ m (e and f), $H = 5000$ m (g and h).
image. This is because the glitter pattern modeled by the Gaussian glitter function contains less energy.

Figure 6 shows the relationships between the correlation function of the intensities of the image (glitter pattern), $C_I(r)$, vs. the correlation function of the surface slopes, $C_M(r)$. The values used in these calculations are $\lambda_{M}^{(30)} = -0.463$, $\sigma_{M}^{(03)} = -0.463$, $\lambda_{M}^{(12)} = -0.10438$, $\lambda_{M}^{(21)} = -0.10438$, and $\sigma_{M} = 0.2121$. It can be observed for $H = 100$ m the higher values are for $50^\circ$ incidence angles, and for $10^\circ$, the inverse process is not possible because the relationship for these two variables is so horizontal. When $H$ increases, it is possible to use any incidence angle from $10^\circ$ to $50^\circ$ in order to go from $C_I(r)$ to $C_M(r)$. Again, like in the variances, the inverse change in the correlations curves, it is observed almost well-defined since 500-m height of the detector. In all the cases, the behavior of the curves is so similar that does not matter if we model the glitter pattern with the Rect or the Gaussian glitter function.

4. Conclusions

The contribution of skewness and kurtosis in these new relationships increase the value of the curves in the vertical axis, but for the correlations function, these values decrease. By the way, it is possible to take into account the inverse process without problem. When the Rect glitter function is used (Equation (4)), it is more visible the changes in the curves of the variance when the contribution of skewness vs. the contribution of skewness and kurtosis is compared (Figures 2(a)–(b), 3(a)–(b), 4(a)–(b), and 5(a)–(b)). For $H = 100$ m and for $\theta_s = 50^\circ$, the variance of the intensities of the image, $\sigma_I^2$, increases for small values of the variance of the slopes, $\sigma_M^2$. For minor angles, this increment in $\sigma_I^2$ is more visible for bigger values of $\sigma_M^2$. When $H = 500$ m, this difference is more marked to incidence angles of $30^\circ$, $20^\circ$, and $10^\circ$. When $H = 1000$ and $5000$ m, this increment is visualized to an incidence angle of $10^\circ$. When the Gaussian glitter function (Equation (5)) is used, it is not possible, visually, to observe changes in the curves (Figures 2(c)–(d), 3(c)–(d), 4(c)–(d), and 5(c)–(d)). This is because the glitter pattern modeled by the Gaussian glitter function contains less energy. However, all these last curves present smaller values for $\sigma_I^2$. In the correlations, it is possible to observe the same behavior of the curves for all the cases.

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References