Solution of Eigenvalue Integral Equation with Exponentially Oscillating Covariance Function

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1. Introduction

In theories of signal detection and filtering [1]–[3], there frequently occur the KL eigenvalue integral equation or the Fredholm equation of the first type whose kernel is the covariance function of the stationary process,

\[ \lambda \varphi(t) = \int_{-T}^{T} K(|t-s|) \varphi(s) \, ds, \quad (1) \]

where \( K(x) \) represents the covariance function of a continuous stationary second order process possessing a continuous spectral density \( S(w) \), \(-T \leq t \leq T\). Denote ensemble average by \( \mathbb{E}[\cdot] \) and let \( m(t) = \mathbb{E}[n(t,v)] \) and \( K(t,s) = \mathbb{E}[(n(t,v)-m(t))(n(s,v)-m(s))] \). Here \( K(t,s) \) is the covariance function of the process \( n(t,v) \). If the covariance function is continuous in square \(-T \leq t, s \leq T\), the process is second-order continuous in \(-T \leq t \leq T\), and if in addition \( K(t,s) = K(|t-s|) \), the one is second-order stationary. From the theorem of Karhunen [3] any second-order stationary random function \( n(t,v) \) which is second-order continuous in \(-T \leq t \leq T\) may be expanded as follows:

\[ n(t,v) = m(t) + \sum_{k=1}^{\infty} \sqrt{\lambda_k} a_k(v) \varphi_k(t), \quad (2) \]

with convergence in the mean for every \( t \) in \([-T,T]\). The quantities \( \lambda_k \) and \( \varphi_k(t) \) are determined from the integral equation (1). Here \( \varphi_k(t) \) are orthonormal over \(-T \leq t \leq T \) and \( \{a_k(v)\} \) are normalized uncorrelated random variables; that is,

\[ E[a_k(v) a_l(v)] = \delta_{kl}, \quad (3) \]

where \( \delta_{kl} \) is the Kronecker delta.

For Gaussian processes the \( \{a_k(v)\} \) are independent normal random variables. These properties have been exploited to great advantage in many theoretical applications. Several techniques have been proposed for solving the KL eigenvalue integral equation when the spectral density of the process is a rational function of frequency [4]–[6]. However, the proposed general methods are computationally costly. They require solving complex systems of transcendental equations. A simple analytical solution has been found for the case of the exponential covariance function [3]. However, the covariance function of real images is often exponentially oscillating,

\[ K(x) = K(|t-s|) = P \exp(-|x|) \cos(\beta x), \quad (4) \]

where \( P \) mean-square value of the zero-mean process, \( \alpha \) and \( \beta \) are nonnegative parameters of correlation and oscillation, respectively. In this case the spectral density can be expressed as

\[ S(w) = \frac{2\alpha (\alpha^2 + \beta^2 + w^2)}{w^4 + 2w^2(\alpha^2 - \beta^2) - (\alpha^2 + \beta^2)^2}, \quad (5) \]

where \( S(w) \) is the spectral density.

2. Solution of Eigenvalue Integral Equation

A basic idea of the method is straightforward. We convert the integral equation to a differential equation whose solution can be easily found. Then we substitute the solution back into the integral equation to satisfy the boundary conditions. First, we substitute (4) into (1) and eliminate the magnitude sign as follows:

\[ \lambda \varphi(t) = \int_{-T}^{T} P \exp(-\alpha (t-s)) \cos(\beta (t-s)) \varphi(s) \, ds \]

\[ + \int_{-T}^{T} P \exp(-\alpha (s-t)) \cos(\beta (s-t)) \varphi(s) \, ds, \quad (6) \]
where \(-T \leq t \leq T\).

Differentiating four times, we have a fourth-order linear differential equation with constant coefficients

\[
\varphi^{(4)}(t) - 2\left(\alpha^2 - \beta^2 - \frac{\alpha P}{\lambda}\right)\varphi^{(2)}(t) + (\alpha^2 + \beta^2)\left(\alpha^2 + \beta^2 - \frac{2\alpha P}{\lambda}\right)\varphi(t) = 0, \quad (7)
\]

for \(\lambda \neq 0\). The integral equation satisfies for the following condition, \(0 < \lambda < 2\alpha/\alpha^2 + \beta^2\) and the roots of the characteristic equation are given by

\[
b_{1,2} = \left(\sqrt{D} - \left(\alpha^2 - \beta^2 - \frac{\alpha}{\lambda}\right)\right),
\]

\[
b_{3,4} = \left(\left(\alpha^2 - \beta^2 - \frac{\alpha}{\lambda}\right) + \sqrt{D}\right), \quad (8)
\]

where \(\lambda = \lambda/P\),

\[
D = \left(\alpha^2 - \beta^2 - \frac{\alpha}{\lambda}\right) - (\alpha^2 + \beta^2)\left(\alpha^2 + \beta^2 - \frac{2\alpha}{\lambda}\right) > 0
\]

and the terms in the brackets on the right-hand side are positive. A general solution of the integral equation can be written as follows:

\[
\varphi(t) = C_1 \exp(ib_1t) + C_2 \exp(-ib_1t) + C_3 \exp(b_3t) + C_4 \exp(-b_3t), \quad (9)
\]

here \(b_2 = -b_1, b_4 = -b_3\).

Substituting (9) into (7) and performing the integration, we obtain

\[
\begin{align*}
\left\{ \begin{array}{l}
C_1 \exp(ib_1T)(\alpha - ib_1) - C_3 \exp(ib_3T)(\alpha + ib_3) + C_2 \exp(-ib_1T)(\alpha + ib_1) - C_4 \exp(-ib_3T)(\alpha - ib_3) \\
+ C_3 \exp(ib_3T)(\alpha - ib_3) + C_4 \exp(-ib_3T)(\alpha + ib_3) - C_1 \exp(ib_1T)(\alpha + ib_1) + C_2 \exp(-ib_1T)(\alpha - ib_1) = 0
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
C_1 \exp(ib_1T)(\alpha - ib_1) - C_3 \exp(ib_3T)(\alpha + ib_3) + C_2 \exp(-ib_1T)(\alpha + ib_1) - C_4 \exp(-ib_3T)(\alpha - ib_3) \\
+ C_3 \exp(ib_3T)(\alpha - ib_3) + C_4 \exp(-ib_3T)(\alpha + ib_3)
\end{align*} = 0
\]

\[
(10)
\]

It can be easily verify that if either (i) \(\{C_1 = C_2, C_3 = C_4\}\) or (ii) \(\{C_1 = -C_2, C_3 = -C_4\}\) (10) satisfies for all time. Eliminating \(\lambda\) from (8) and combining equations (8) and (10), for these two conditions we obtain the following system of equations:

\[
\begin{align*}
C_1^\pm \left( \frac{\exp(ib_1T)(\alpha - ib_1)}{(\alpha - ib_1)^2 + \beta^2} \pm \frac{\exp(-ib_1T)(\alpha + ib_1)}{(\alpha + ib_1)^2 + \beta^2} \right) \\
+ C_3^\pm \left( \frac{\exp(ib_3T)(\alpha - ib_3)}{(\alpha - ib_3)^2 + \beta^2} \pm \frac{\exp(-ib_3T)(\alpha + ib_3)}{(\alpha + ib_3)^2 + \beta^2} \right) = 0
\end{align*}
\]

\[
\begin{align*}
C_1^\pm \left( \frac{\exp(ib_1T)(\alpha - ib_1)}{(\alpha - ib_1)^2 + \beta^2} \pm \frac{\exp(-ib_1T)(\alpha + ib_1)}{(\alpha + ib_1)^2 + \beta^2} \right) \\
+ C_3^\pm \left( \frac{\exp(ib_3T)(\alpha - ib_3)}{(\alpha - ib_3)^2 + \beta^2} \pm \frac{\exp(-ib_3T)(\alpha + ib_3)}{(\alpha + ib_3)^2 + \beta^2} \right) = 0
\end{align*} = 0
\]

\[
\begin{align*}
b_2^2 &= (\alpha^2 + \beta^2)\left(\alpha^2 - 3\beta^2 + b_3^2\right) \\
b_3^2 &= (\alpha^2 + \beta^2)\left(\alpha^2 - \beta^2 + b_3^2\right)
\end{align*}
\]

where “+” is used for the condition (i), and “−” is used for the condition (ii). Equation (11) can be simplified as follows:

\[
\begin{align*}
&b_1 \left( R \sin b_1 T + I \cos b_1 T \right) \left( R \cosh b_3 T + I \sinh b_3 T \right) \\
&- b_3 \left( R \sin b_1 T + I \cos b_1 T \right) \cdot \left( R \cosh b_3 T + I \sinh b_3 T \right) = 0,
\end{align*}
\]

for (i)

\[
\begin{align*}
&b_1 \left( R \cos b_1 T - I \sin b_1 T \right) \left( R \sinh b_3 T + I \cosh b_3 T \right) \\
&- b_3 \left( R \cos b_1 T + I \sin b_1 T \right) \cdot \left( R \cosh b_3 T + I \sinh b_3 T \right) = 0,
\end{align*}
\]

for (ii)

\[
\begin{align*}
b_3^2 &= (\alpha^2 + \beta^2)\left(\alpha^2 - 3\beta^2 + b_3^2\right) \left(\alpha^2 - \beta^2 + b_3^2\right)
\end{align*}
\]

\[
\begin{align*}
&b_2^2 = (\alpha^2 + \beta^2)\left(\alpha^2 - \beta^2 + b_3^2\right) \left(\alpha^2 - 3\beta^2 + b_3^2\right)
\end{align*}
\]

\[
(12)
\]

where \(R = \alpha^2 + \beta^2 - b_2^2, I = 2ab_1\).

From the second equation of the system in (11), \(C_3^\pm\) as a function of \(C_1^\pm\) can be written as

\[
C_3^\pm = -C_1^\pm Q^\pm, \quad (13)
\]

with

\[
\begin{align*}
Q^+ &= \frac{\bar{R}^2 - \bar{I}^2}{R^2 + \bar{I}^2} R \cos b_1 T - I \sin b_1 T, \quad (\text{for (i)})
\end{align*}
\]

\[
\begin{align*}
Q^- &= \frac{\bar{R}^2 - \bar{I}^2}{R^2 + \bar{I}^2} R \sin b_1 T + I \cos b_1 T, \quad (\text{for (ii)})
\end{align*}
\]

\[
(14)
\]

Solving the system in (12), two sets of \(\{b_1^\pm(k), b_3^\pm(k)\}, \{b_1^\pm(k), b_3^\pm(k)\}, k = 1, 2, \ldots\) can be numerically calculated. Finally, the eigenfunctions for the conditions (i) and (ii) are given by

\[
\varphi_k(t) = \begin{cases} 
C_1^+ \left( k \sin b_1^+ (k) t \right) + C_3^+ \left( k \cos b_3^+ (k) t \right), \\
C_1^- \left( k \sin b_1^- (k) t \right) + C_3^- \left( k \cos b_3^- (k) t \right), \\
-T \leq t \leq T, \quad k = 1, 2, \ldots
\end{cases}
\]

\[
(15)
\]

The expressions for the normalizing constants \(C_1^\pm(k)\) are useless for practical work; however, we derive them using the property of orthonormality of the eigenfunctions (see (3)) to compare the results with the ones obtained for the exponential covariance function [3]. They are written as

\[
\begin{align*}
C_1^+ &= \left\{ \frac{T}{2} \left[ 1 + \sin \frac{2b_1^+ T}{b_1^+} + \left( Q^+ \right)^2 \sinh \frac{2b_3^+ T}{b_3^+} \right] \right\}^{-1/2} \\
&- 4Q^+ \left( b_1^+ \sin b_1^+ T \cosh b_3^+ T + b_3^+ \cos b_3^+ T \sinh b_3^+ T \right) b_1^2 + b_3^2
\end{align*}
\]

\[
\begin{align*}
C_1^- &= \left\{ \frac{T}{2} \left[ 1 - \sin \frac{2b_1^- T}{b_1^-} - \left( Q^- \right)^2 \sinh \frac{2b_3^+ T}{b_3^-} \right] \right\}^{-1/2} \\
&+ 4Q^- \left( b_1^- \cos b_1^- T \sinh b_3^- T - b_3^- \sin b_3^- T \cosh b_3^- T \right) b_1^2 + b_3^2
\end{align*}
\]

\[
(16)
\]
here for simplicity we omit the index \( k \). The coefficients \( C^\pm_3 \) can be calculated from (13). Let us sort the solutions \( \{ b^+_1(k) \} \) and \( \{ b^-_1(k) \} \) in ascending order with respect to their values as follows: \( b_1(1) < b_1(2) < b_1(3) < \ldots \) and \( b_3(1) < b_3(2) < b_3(3) < \ldots, k = 1, 2, \ldots \). The odd-numbered solutions correspond to the condition (i), whereas the even-numbered solutions correspond to the condition (ii). The corresponding eigenvalues are calculated as

\[
\lambda_k = \frac{2P\alpha (\alpha^2 + \beta^2 + b^2_1(k))}{b^2_1(k) + 2b^2_1(k)(\alpha^2 - \beta^2) + (\alpha^2 + \beta^2)^2}, \quad (17)
\]

If \( \beta = 0 \) we arrive to known results for the exponential covariance function. In this case \( C^\pm_3 = 0, Q^\pm = 0 \) and the expressions for eigenfunctions are simplified to

\[
\varphi_k(t) = \begin{cases} 
C^+_1(k) \cos \left( b^+_1(k) t \right) \\
C^-_1(k) \sin \left( b^-_1(k) t \right) 
\end{cases}, \quad -T \leq t \leq T, \quad k = 1, 2, \ldots \quad (18)
\]

with normalizing coefficients

\[
C^+_1(k) = \left( T \left( 1 \pm \frac{\sin 2b^+_1(k) T}{2b^+_1(k) T} \right) \right)^{-1/2}. \quad (19)
\]

The corresponding eigenvalues are calculated as

\[
\lambda_k = \frac{2P\alpha}{\alpha^2 + b^2_1(k)}, \quad k = 1, 2, \ldots \quad (20)
\]

3. Conclusion

An analytical solution of the Karhunen-Loeve integral equation for a practical case when the covariance function of a stationary process is exponentially oscillating has been presented. The explicit expressions for the eigenfunctions and eigenvalues have been obtained. The eigenfunctions can be used for optimal series expansions of random processes in image processing, speech processing, pattern recognition, communication systems and generalized filtering.

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References