First and second order statistics of rough random surfaces from remote sensing images considering a Gaussian glitter function

MARK MARÍN-HERNÁNDEZ and JOSUÉ ALVAREZ-BORREGO*

Centro de Investigación Científica y de Educación Superior de Ensenada, División de Física Aplicada, Departamento de Optica, Km. 107 carretera Tijuana-Ensenada, Ensenada, B.C., México

(Received 9 March 1998; revision received 24 June 1998)

Abstract. The problem of obtaining information about the statistical properties of rough random surfaces from remote sensing images was solved by Alvarez-Borrego [1993, PhD thesis, CICESE, Ensenada B.C., México], who presented a model for one- and two-dimensional surfaces with numerical examples and experimental results. In this paper the same idea is followed, using the same geometrical parameters, and also for one- and two-dimensional surfaces. However, a Gaussian glitter function rather than the rect or circ types is used.

This work was motivated by the opinion of some investigators who also work in the same field and who have commented that possibly a simpler manner to describe the first and second order statistics of rough random surfaces exists using a Gaussian glitter function and not the rect or circ functions.

The results show that with a Gaussian glitter function it is possible to obtain an analytical relation in the equations and not only a numerical one as was found using the circ or rect functions. But upon the analysis of the behaviour of the analytical relations obtained in different geometries it was observed that the values obtained for the variance and for correlations were quite poor. Also, the image obtained using the Gaussian glitter function left us with the initial problem of having a range of grey values in the image, which is a difficulty when it is desired to apply this model to real physical situations. We conclude that the rect and circ functions are of higher utility in the application of the model than the Gaussian glitter function.

1. Introduction

The physical studies of the marine environment, especially in the coastal zone, constitute a large part of the knowledge of the hydrodynamics of these areas. Studies with the goal of understanding the details of the hydrodynamics of coastal waters are of particular importance for meeting the needs presented during the exploitation of natural resources, and in the design of projects and installations that will be utilized by industry, tourism and mariculture.

Oceanographers have attempted to understand the complex structure of the marine surface to develop techniques that will permit obtaining time series of wave

*Address for correspondence: Dept. Optics, Applied Physics Division, CICESE, PO Box 434944, San Ysidro (San Diego), CA 92143-4944. USA; e-mail: josue@cicese.mx.
heights with the help of pressure sensors. The importance of obtaining this information is that it allows the explanation of the generation, propagation and prediction of waves, as well as the interchange of energy between water masses.

Recently, with the great need to be able to count on local measurements of wave actions on the spatial and temporal scale, studies to estimate wave action in a quantitative manner have been employed using radar images and aerial photographs. Since the end of the past century, many optical parameters had begun to be measured (ascending and descending irradiance, etc.), as well as oceanographic parameters (temperature, salinity, density, etc.) with the goal to study the sea surface [1]. The irradiance of the light reflected from the sea surface was the first optical parameter that was measured. Then, the spectral irradiance at different depths depending on other oceanographic variables was utilized to determine the organic primary productivity of the ocean. The advantages that the optical methods offer for the study of the marine surface are that they can be applied for the study of waves in both deep and shallow waters, and that the processing of data is rapid and simple compared to other methods [2].

At the middle of this century, Cox and Munik [3, 4] studied the slopes of the sea surface derived from the spectral reflections of light from the marine surface and the glitter pattern, both of which are recorded in aerial photographs. In a first approximation, the authors found that the distribution of the slopes was Gaussian. Simultaneously with these studies, Barber [5] estimated the directionality of the ocean waves using the diffraction pattern of an image of the sea surface.

Later, in 1958, Cox [6] measured the slopes of capillary waves, and found a quite complicated behaviour of the variance of the slopes as a function of the wind velocity. Further investigations into the study of wave action were made by Stilwell [7], who analysed photographs of the sea surface with the help of an optical system; by Kasevich [8], who continued Stilwell’s work, and concluded that the light reaching the camera after the reflection off the sea surface is related to the orientation and height of the camera; and by Peppers and Ostrem [9], who proposed a model to determine the slopes of the waves using photographs of the sea.

Actually, the statistical properties of the sea surface obtained from aerial photographs were calculated. Alvarez-Borrego [2, 10] worked with rough random surfaces in one and two dimensions, applying a function, named the glitter function, to find the relation between the autocorrelation function of the variations of the intensities in the image and the autocorrelation function of the surface heights.

The glitter function describes the glare or glitter produced when the light provided by the source is reflected only once off the surface. In this way, it can be observed that the image of the surface consists of light and dark regions forming a pattern of glitter, which is coded in two levels by choosing an adequate threshold after analysing the histogram of image intensity. To the brilliant levels, a value of 1 is assigned; and to the dark regions a value of 0 is assigned. In this way he found that the pattern of glitter obtained in the image plane contains statistical information on the heights of the sea surface and, depending on the local slope, the light reflected from each point of the surface contributes or does not contribute to the image.

In this paper, the methodology developed by Alvarez-Borrego [2] is presented, but for a different glitter function. The function utilized here is a Gaussian
function. This function physically describes the profile of the intensity of source of light that has a Gaussian form rather than a constant distribution, which was the case in the situation presented before.

This work was motivated by the opinion of some investigators [11] who also work in the same field and who have commented that possibly a simpler manner to describe the first and second order statistics of rough random surfaces exists using a Gaussian glitter function.

The goal in this paper is to obtain the moments of first and second order of random rough surfaces and to compare the results with the statistics obtained when a rect or circ glitter function is used.

2. **Geometry of the problem**

The physical situation is shown in figure 1. The surface is illuminated by a Gaussian and incoherent source $\sigma$ of limited angular extent, with wavelength $\lambda$. Its image is formed in $D$ by an aberration-free optical system. In figure 1, $\theta_s$, represents the mean angle subtended by the source $\sigma$ and the normal to the mean surface, while $\theta_d$ represents the mean angle subtended by the optical system of the detector with the normal to the mean surface. The apparent diameter of the source is $\beta$. Light from the source is reflected from the surface just once and, depending on the slope, the light reflected will or will not be part of the image. In broad terms, the image consists of bright and dark regions that we call a glitter pattern. The patterns obtained in the image contain statistical information about the surface [2]. Our analysis involves three random processes: the surface profile, surface slopes, and image.

In order to obtain statistical information of the height variations of a rough surface, certain variables must first be defined. The variations of heights are given by the function $\zeta(x)$, which in all cases represents a random Gaussian process with a correlation function $\sigma_z^2 C_\zeta(\tau) = \langle \zeta(x + \tau)\zeta(x) \rangle$ and a variance $\sigma_z^2$. The slope of the surface is represented by $\Pi(x)$, with a correlation function $\sigma_{\Pi}^2 C_{\Pi} = \langle \Pi(x + \tau)\Pi(x) \rangle$ and variance $\sigma_{\Pi}^2$.

The correlation function of the heights of the surface $C_\zeta(\tau)$ is related to the correlation function of the slopes $C_{\Pi}(\tau)$, as [12]:

$$\sigma_{\Pi}^2 C_{\Pi}(\tau) = -\frac{d^2 \sigma_z^2 C_\zeta(\tau)}{d\tau^2}. \quad (1)$$

![Figure 1. Physical situation.](image)
However, the slopes of the surface, $\Pi(x)$, are a random Gaussian process, and the probability density function, for two distinct positions $x_1$ and $x_2$ is given by [13]:

$$p(\Pi_1, \Pi_2) = \frac{1}{2\pi \sigma_{\Pi}^2} \exp \left[ -\frac{\Pi_1^2 + \Pi_2^2 - 2C_{\Pi}(\tau)\Pi_1\Pi_2}{2\sigma_{\Pi}^2[1 - C_{\Pi}(\tau)]} \right],$$  \hspace{1cm} (2)

where, $\Pi_1$ and $\Pi_2$ correspond to $\Pi(x_1)$ and $\Pi(x_2)$, respectively, and $\tau = x_1 - x_2$.

The image, $i(x)$, formed in plane D (figure 1) is a nonstationary random process determined by $\Pi(x)$ and the geometry of the problem. The process $i(x)$ is not stationary even though the processes $\zeta(x)$ and $\Pi(x)$ are stationary, because the statistical properties of $i(x)$ are a function of their position in $x$.

This nonstationary behaviour can be avoided if we consider a plane of observation that is sufficiently distant, and restrict ourselves to the field of vision. In this case and only with the previous considerations can we have a correlation function of the process $i(x)$, associated with the image, as $\sigma_i^2 C_i(\tau) = \langle i(x + \tau) i(x) \rangle$, and a variance of $\sigma_i^2$.

The glitter pattern, bright regions and dark regions, that the image forms is determined by the surface and by the ‘glitter function’, calculated from the positions and the apparent diameters of the source and detector. This function operates over the slopes and is defined as $B(\Pi(x))$. With this function we can find the spatial location and the extent of the brilliance. In this way the autocorrelation of the image, $\sigma_i^2 C_i(\tau)$, can be described by:

$$\sigma_i^2 C_i = \langle B(\Pi_1) B(\Pi_2) \rangle,$$  \hspace{1cm} (3)

or

$$\sigma_i^2 C_i(\tau) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{B(\Pi_1) B(\Pi_2)}{2\pi \sigma_{\Pi}^2[1 - C_{\Pi}(\tau)]^{1/2}} \exp \left[ -\frac{\Pi_1^2 + \Pi_2^2 - 2C_{\Pi}(\tau)\Pi_1\Pi_2}{2\sigma_{\Pi}^2[1 - C_{\Pi}(\tau)]} \right] d\Pi_1 d\Pi_2.$$  \hspace{1cm} (4)

The methodology used in this work differs from that used by Alvarez-Borrego [2], in the type of glitter function assumed. He used a glitter function rect($x$), assuming that the intensity of the incoming light was uniform over the range of slopes. However, in this work a glitter function of Gaussian type is considered such that the intensity of light is greater in the centre of the range of slopes and diminishes towards its edges.

The difference between the two functions can be seen graphically in figure 2, where $2a$ corresponds to the width of the source.

3. **Relations among the variances of the intensities in the image, slopes and surface heights**

3.1. **One dimension**

The glitter function in the Gaussian form can be defined as:

$$B(\Pi) = \exp \left[ -\frac{(\Pi - \Pi_0)^2}{a^2} \right],$$  \hspace{1cm} (5)
where $\Pi = \tan \alpha$, and $\alpha$ is the angle between the $x$-axis and the tangent plane to the surface at the point $(x_0, \zeta(x_0))$, $\Pi_0 = \tan \left( \frac{\theta_s - \theta_0}{2} \right)$, and $2a = (1 + \Pi_0^2)\beta/2$ defines the width of the source. The glitter function is described by the formation of bright patterns that constitute the image. This function relates the slopes of the surface with the patterns of brilliance that they form. From the information obtained from the glitter pattern we wish to obtain information about the energy spectrum or the correlation function of the heights of the surface.

From the theory of random processes, we know that the random process $\Pi(x)$ representing the slopes of the surface is a random Gaussian process. Therefore in one dimension its probability density function is given by:

$$p(\Pi) = \frac{1}{\sigma_{\Pi} \sqrt{2\pi}} \exp \left[ -\frac{\Pi^2}{2\sigma_{\Pi}^2} \right],$$

where $\sigma_{\Pi}^2$ is the variance of the slopes. The mean of the image, $\mu_i$, may be written [10]

$$\mu_i = \int_{-\infty}^{\infty} B(\Pi) p(\Pi) \, d\Pi.$$  

Substituting equations (5) and (6) in equation (7),

$$\mu_i = \frac{1}{\sigma_{\Pi} \sqrt{2\pi}} \int_{-\infty}^{\infty} \exp \left[ -\frac{(\Pi - \Pi_0)^2}{a^2} \right] \exp \left[ -\frac{\Pi^2}{2\sigma_{\Pi}^2} \right] \, d\Pi,$$

and integrating, we obtain:

$$\mu_i = -\frac{1}{\left( 1 + 2\frac{\sigma_{\Pi}^2}{a^2} \right)^{1/2}} \exp \left[ -\frac{\Pi_0^2}{a^2 + 2\sigma_{\Pi}^2} \right].$$

The variance of the intensities in the image, $\sigma_i^2$, is defined as [10]:

$$\sigma_i^2 = \int_{-\infty}^{\infty} [B(\Pi) - \mu_i]^2 p(\Pi) \, d\Pi,$$

and substituting equations (5), (6) and (9) in (10), and integrating, we obtain:

$$\sigma_i^2 = \frac{1}{\left( 1 + 4\frac{\sigma_{\Pi}^2}{a^2} \right)^{1/2}} \exp \left[ -\frac{2\Pi_0^2}{a^2 + 4\sigma_{\Pi}^2} \right] - \frac{1}{1 + 2\frac{\sigma_{\Pi}^2}{a^2}} \exp \left[ -\frac{2\Pi_0^2}{a^2 + 2\sigma_{\Pi}^2} \right].$$

Figure 2. Rect and Gaussian functions in one dimension.
The relation that exists between the variance of the intensities in the image, $\sigma_i^2$, and the variance of the slopes of the surface, $\sigma_P^2$, is represented in equation (11) which can be seen graphically in figure 3(a), for some typical cases, utilizing the geometry described in figure 1 with $\theta_d = 0$ and the incident angle, $\theta_s$, varying from $10^\circ$ to $50^\circ$, and $\beta = 0.68^\circ$. Along the horizontal axis we have the variance of the surface slopes, $\sigma_P^2$, and along the vertical axis we have the variance of the intensities in the image, $\sigma_i^2$. In the figure we can observe the dependence of this relationship on the angular position of the source, $\theta_s$. From this graph, we also can observe that for small incidence angles ($0^\circ$ to $10^\circ$) and small values of the variance of the surface slopes, it is possible to obtain larger values of the variance of the intensities in the image. From equation (11), we can see that this behaviour is independent of any surface height power spectrum that we are analysing, because this relation depends on the probability density function of the surface slopes and the geometry of the experiment only.

From figure 3(a) we can also see that for some angles of incidence, $\theta_s$, at the same value of image intensity variance, two values of slope variance, $\sigma_P^2$, exist. This ambiguity can be resolved by analysing the images that correspond to two or more incidence angles and selecting a value of the slope variance which is consistent with all these data [2]. If we compare the graph obtained for this case with the one obtained by Alvarez-Borrego [2, 10], figure 3(b), we can observe that the behaviour of the curves is similar; however, the values found for the Gaussian function are smaller than for the rect type function.

**Figure 3.** Relationship between the variance of the surface slopes with the variance of the intensities in the image. One-dimensional case. (a) Using a Gaussian glitter function. (b) Using a rect glitter function.
The relationship between \( \sigma_{II}^2 \) and \( \sigma_{\zeta}^2 \) can be derived from equation (1).

3.2. Two-dimensional case

In the same way as in the previous case, the glitter function and probability density function in two dimensions are defined respectively as:

\[
B(\Pi_x, \Pi_y) = \exp \left[ \frac{(\Pi_x - \Pi_0)^2 + (\Pi_y - \Pi_0)^2}{a^2} \right],
\]

\[
p(\Pi_x, \Pi_y) = \frac{1}{2\pi \sigma_{II} \sigma_{II_y}} \exp \left[ -\frac{\Pi_x^2}{2\sigma_{II}^2} - \frac{\Pi_y^2}{2\sigma_{II_y}^2} \right].
\]

The mean of the image, \( \mu_i \), is defined as:

\[
\mu_i = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} B(\Pi_x, \Pi_y) p(\Pi_x, \Pi_y) \, d\Pi_x \, d\Pi_y.
\]

Substituting equations (12) and (13) in (14), we obtain

\[
\mu_i = \frac{1}{2\pi \sigma_{II} \sigma_{II_y}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp \left[ -\frac{(\Pi_x - \Pi_0)^2 + (\Pi_y - \Pi_0)^2}{a^2} \right]
\times \exp \left[ -\frac{\Pi_x^2}{2\sigma_{II}^2} - \frac{\Pi_y^2}{2\sigma_{II_y}^2} \right] \, d\Pi_x \, d\Pi_y.
\]

3.2.1. Isotropic Case

In this case \( \sigma_{II_x} = \sigma_{II_y} = \sigma_{II} \), so equation (15) may be written as

\[
\mu_i = \frac{1}{1 + 2\frac{\sigma_{II}^2}{a^2}} \exp \left[ -\frac{2\Pi_0^2}{a^2 + 2\sigma_{II}^2} \right].
\]

Defining the variance as:

\[
\sigma_i^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [B(\Pi_x, \Pi_y) - \mu_i]^2 p(\Pi_x, \Pi_y) \, d\Pi_x \, d\Pi_y,
\]

we find that

\[
\sigma_i^2 = \frac{1}{1 + 4\frac{\sigma_{II}^2}{a^2}} \exp \left[ -\frac{4\Pi_0^2}{a^2 + 4\sigma_{II}^2} \right] - \frac{1}{\left(1 + 2\frac{\sigma_{II}^2}{a^2}\right)^2} \exp \left[ -\frac{4\Pi_0^2}{a^2 + 2\sigma_{II}^2} \right].
\]

Figure 4(a) shows the relationship between the variance of the intensities in the image, \( \sigma_i^2 \), and the variance of the surface slopes, \( \sigma_{\zeta}^2 \). In this case the angles of incidence were varied from 10 to 50° and the angle of detection, \( \theta_d \), was maintained at 0°. The horizontal axis gives the variance of the surface slopes and the vertical axis gives the variance of the intensities in the image. For this case we observe a behaviour similar to the one-dimensional case where small incidence angles and low slope variances gave large values of the variance of the image intensity.

For this case, the curves are more abrupt than those obtained for the one-dimensional case and we can see that for large values of \( \sigma_{II}^2 \) the difference between
the values of $\sigma_i^2$ is very small. In this case the results obtained with the circ function are shown in Figure 4(b), where the values obtained are larger, and the curves calculated with this function cross.

3.2.2. Anisotropic case

For this case equation (15) may be written as

$$\mu_i = \frac{a^2}{(a^2 + 2\sigma_{ii}^2)^{1/2}(a^2 + 2\sigma_{ij}^2)^{1/2}} \exp \left\{ \left[ \left( \frac{\sigma_{xx}^2}{a^2 + 2\sigma_{xx}^2} + \frac{\sigma_{yy}^2}{a^2 + 2\sigma_{yy}^2} \right) - 1 \right] \left( \frac{\sqrt{2\Pi_0}}{a} \right)^2 \right\}. $$

(19)

The variance is defined as

$$\sigma_i^2 = \frac{a^2}{(a^2 + 4\sigma_{ii}^2)^{1/2}(a^2 + 4\sigma_{ij}^2)^{1/2}} \exp \left\{ \left[ \left( \frac{2\sigma_{xx}^2}{a^2 + 4\sigma_{xx}^2} + \frac{2\sigma_{yy}^2}{a^2 + 4\sigma_{yy}^2} \right) - 1 \right] \left( \frac{2\Pi_0}{a} \right)^2 \right\}
- \left( \frac{a^2}{(a^2 + 2\sigma_{ii}^2)^{1/2}(a^2 + 2\sigma_{ij}^2)^{1/2}} \exp \left\{ \left[ \left( \frac{\sigma_{xx}^2}{a^2 + 2\sigma_{xx}^2} + \frac{\sigma_{yy}^2}{a^2 + 2\sigma_{yy}^2} \right) - 1 \right] \left( \frac{\sqrt{2\Pi_0}}{a} \right)^2 \right\} \right)^2. $$

(20)
Figure 5. Relationship between the variance of the surface slopes along the \( x \)-direction with the variance of the intensities in the image. Anisotropic case. (a) Variance value of the slopes along the \( y \)-direction of 0.02. (b) Variance value of the slopes along the \( y \)-direction of 0.12. Gaussian glitter function case.

Figure 5 shows the relationship between the variance of the intensities in the image and the variance of the surface slopes at different light incidence angles. This figure shows two particular examples where we have fixed the value of \( \sigma_{y}^2 \), and we show the variation of \( \sigma_{i}^2 \) with \( \sigma_{x}^2 \). In figure 5(a) \( \sigma_{y}^2 = 0.02 \), and we can see that for angles greater than 30° the values obtained for \( \sigma_{i}^2 \) are very small and are not even visible in the graph. We can also see that comparing the scale of the two graphs, the scale of \( \sigma_{i}^2 \) in figure 5(a) is much greater than that of the second example. Another aspect that we can see for this case is that the curves described are much smoother and less abrupt than for the previous cases. In the second example \( \sigma_{y}^2 \) was fixed at \( \sigma_{y}^2 = 0.12 \) [figure 5(b)]. In this example we can see that only the curve corresponding to the angle of incidence of 10° appears, and that for angles in the range (20–50°) the values of \( \sigma_{i}^2 \) are too small and cannot be seen in the curves.

If we compare these results with those obtained with the circt function in figure 6, we see, in these graphs, five curves corresponding to different angles appear, and we can also see that for the case \( \sigma_{y}^2 = 0.12 \) there is a crossing of the curves.
4. Relations between the correlation functions of the intensities in the image and of the surface heights

4.1. One-dimensional case

As mentioned before, our analysis involves three random processes: the surface profile, \( \zeta(x) \), its surface slopes, \( \Pi(x) \) and the image, \( i(x) \). Each process has a correlation function, and it was shown in [2] that these three functions are related. The relationship between the correlation functions of the surface heights, \( C_\zeta(\tau) \), and the surface slopes, \( C_\Pi(\tau) \), is given by equation (1), and the relationship between \( C_\Pi(\tau) \) and the correlation function of the intensities in the image, \( C_i(\tau) \), is given by [2]

\[
\sigma_i^2 C(\tau) = \int_{-\infty}^{\infty} B(\Pi_1) B(\Pi_2) p(\Pi_1, \Pi_2) \, d\Pi_1 \, d\Pi_2. 
\] (21)

Using equation (2) and equation (5) we can write

\[
\sigma_i^2 C(\tau) = \frac{1}{2\pi\sigma_i^2(1 - C_i^2(\tau))^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left[ \frac{(\Pi_1 - \Pi_0)^2}{a^2} \right] \exp \left[ \frac{(\Pi_2 - \Pi_0)^2}{a^2} \right] \exp \left[ -\frac{\Pi_1^2 + \Pi_2^2 - 2C(\tau)\Pi_1\Pi_2}{2\sigma_i^2(1 - C_i^2(\tau))} \right] \, d\Pi_1 \, d\Pi_2. 
\] (22)
Defining new variables $X_0 \ldots X_5$:

$$X_0 = \left( \frac{1}{a^2} + \frac{1}{2\sigma_t^2(1 - C_{II}(\tau))} \right),$$

$$X_1 = \left( \frac{C^2_{II}(\tau)}{\sigma_t^4(1 - C_{II}^2(\tau))^2} \right),$$

$$X_2 = \left( X_0 - \frac{X_1}{4X_0} \right),$$

$$X_3 = \left( \frac{\Pi_0 C_{II}(\tau)}{a^2\sigma_t^2(1 - C_{II}^2(\tau))} \right),$$

$$X_4 = \left( \frac{4\Pi_0^2}{a^4} + \frac{4\Pi_0}{a^2} \frac{X_3}{X_0} \right),$$

$$X_5 = \left( \frac{2\Pi_0^2}{a^2} - \frac{\Pi_0^2}{a^4} X_0 \right),$$

we can write the result as:

$$\sigma_t^2 C(\tau) = \frac{1}{\left[ 1 + 4\sigma_t^2(1 - C_{II}(\tau)) \right]^{1/2}} \exp \left[ \frac{X_4 + \left( \frac{X_3}{X_0} \right)^2}{4X_2} - X_5 \right]. \quad (23)$$

In order to achieve the inverse process, using equations (1) and (23), these two equations must satisfy certain conditions. For example, it is required that there exists a one-to-one correspondence among the quantities involved. Equation (23) shows that with a Gaussian glitter function it is possible to obtain an analytical relation and not only a numerical one, as was found using the rect function [10].

In figure 7 we can see the relations between the autocorrelation function of the slopes and the autocorrelation function of the intensities in the image for two different values of $\sigma_{II}$. For the two cases the angle $\theta_d$ is set at $0^\circ$. For the first case, $\sigma_{II} = 0.122$, we observe that for angles of incidence smaller than $10^\circ$ we obtain certain values of $C_t$ corresponding to two values of $C_{II}$. Of course these relationships can be changed, using the same value for the incidence angles, $\theta_s$, if we change the detector angular position, $\theta_d$. We can observe, however, in this example, that for $\theta_s$ values greater than $10^\circ$ the inverse process is possible.

In the second case, $\sigma_{II} = 0.244$, we observe the same behaviour for angles smaller than $15^\circ$, only in this case the described curves are found closer together than in the preceding case. If we compare these results with those found for the rect function (figure 8), we see that the behaviour is similar except that in the examples the described curves are closer together. The $C_t$ and $C_{II}$ relationship depends critically on the value of $\sigma_{II}$ (figures 7 and 8). It is thus very important to know the variance of the slopes, $\sigma_t^2$, with very good accuracy: in this way we can choose good values for $\theta_s$ and $\theta_d$ and achieve the inverse process.
Figure 7. Relationship between the correlation function of the surface slopes and the correlation function of the intensities in the image. Gaussian glitter function case.

Figure 8. Relationship between the correlation function of the surface slopes and the correlation function of the intensities in the image. Rect glitter function case.
4.2. Two-dimensional case

The relationships between the correlation of the surface heights, $C_z(\xi, \eta)$, and the correlations of the surface slopes, $C_{i\ell}(\xi, \eta)$, are given by [10]

$$
\sigma_{i\ell x}^2 C_{i\ell}^x(\xi, \eta) = -\frac{\partial^2 \sigma_z^2 C_z(\xi, \eta)}{\partial \xi^2}, \quad (24a)
$$

$$
\sigma_{i\ell y}^2 C_{i\ell}^y(\xi, \eta) = -\frac{\partial^2 \sigma_z^2 C_z(\xi, \eta)}{\partial \eta^2}, \quad (24b)
$$

$$
\sigma_{i\ell x} \sigma_{i\ell y} C_{i\ell}^{xy}(\xi, \eta) = -\frac{\partial^2 \sigma_z^2 C_z(\xi, \eta)}{\partial \xi \partial \eta}. \quad (24c)
$$

The relationship between the correlation functions of the surface slopes, $C_{i\ell}(\xi, \eta)$, and the intensities in the image, $C_i(\xi, \eta)$, is given by

$$
\sigma_i^2 C_i(\xi, \eta) = \iint B(\Pi_x, \Pi_y) B(\Pi_{x'}, \Pi_{y'}) p(\Pi_x, \Pi_y, \Pi_{x'}, \Pi_{y'}) \, d\Pi_x \, d\Pi_y \, d\Pi_{x'} \, d\Pi_{y'}, \quad (25)
$$

where $B(\Pi_x, \Pi_y)$ is defined by equation (12) and $p(\Pi_x, \Pi_y, \Pi_{x'}, \Pi_{y'})$ is the two-dimensional probability density function for two different positions

$$
p(\Pi_x, \Pi_y, \Pi_{x'}, \Pi_{y'}) = \frac{1}{4\pi^2|\Delta|^{1/2}} \exp \left[ -\frac{1}{2|\Delta|} \sum_{n=1}^4 \sum_{m=1}^4 |\Delta|_{nm} \Pi_n \Pi_m \right], \quad (26)
$$

considering here that:

$$
\Pi_1 = \Pi_x = \Pi_x(x_1),
$$

$$
\Pi_2 = \Pi_y = \Pi_y(y_1),
$$

$$
\Pi_3 = \Pi_{x'} = \Pi_x(x_2),
$$

$$
\Pi_4 = \Pi_{y'} = \Pi_y(y_2).
$$

In this case, we assume that the probability density function is Gaussian. $C_{i\ell}(\xi, \eta)$ is inside the probability density function represented by equation (26) [12].

The argument of the exponential is then:

$$
\arg = \frac{1}{2|\Delta|} \sum_{n=1}^4 \sum_{m=1}^4 |\Delta|_{nm} \Pi_n \Pi_m = \frac{1}{2|\Delta|} \left[ |\Delta|_{11}(\Pi_x^2 + \Pi_{x'}^2) + |\Delta|_{22}(\Pi_y^2 + \Pi_{y'}^2) \right]
$$

$$
+ 2|\Delta|_{13} \Pi_x \Pi_{x'} + 2|\Delta|_{14} \Pi_x \Pi_{y'} + 2|\Delta|_{24} \Pi_y \Pi_{x'} + 2|\Delta|_{12} (\Pi_x \Pi_y + \Pi_{x'} \Pi_{y'})
$$

$$
+ 2|\Delta|_{14} (\Pi_x \Pi_{y'} + \Pi_{x'} \Pi_y) \right],
$$

where, for the isotropic case, $\sigma_{i\ell x}^2 = \sigma_{i\ell y}^2 = \sigma_{i\ell}^2$. 
\[ |\Delta|_{11} = \sigma_{II}^6 - \sigma_{II}^2 C_{IIy} + 2C_{IIX} [C_{IIy} - \sigma_{II}^3], \]
\[ |\Delta|_{22} = \sigma_{II}^6 - \sigma_{II}^2 C_{IIx} + 2C_{IIX} [C_{IIx} - \sigma_{II}^3], \]
\[ |\Delta|_{13} = 2C_{IIX} [\sigma_{II}^2 - C_{IIy}] - C_{IIx} [\sigma_{II}^4 - \sigma_{II}^2], \]
\[ |\Delta|_{24} = 2C_{IIX} [\sigma_{II}^2 - C_{IIx}] - C_{IIy} [\sigma_{II}^4 - \sigma_{II}^2], \]
\[ |\Delta|_{12} = C_{IIX} [\sigma_{II}^4 - \sigma_{II}^2 C_{IIy} - \sigma_{II}^2 C_{IIy} + C_{IIx} C_{IIy}], \]
\[ |\Delta|_{14} = C_{IIX} [\sigma_{II}^4 - \sigma_{II}^2 C_{IIy} - \sigma_{II}^2 C_{IIx} + C_{IIx} C_{IIy}], \]
\[ |\Delta| = \sigma_{II}^8 - \sigma_{II}^4 C_{IIy} - \sigma_{II}^4 C_{IIx}^2 + C_{IIx}^2 - 2\sigma_{II}^2 \sigma_{II}^6 C_{IIy} - 4C_{IIX} (\sigma_{II}^4 - \sigma_{II}^2 C_{IIy} - \sigma_{II}^2 C_{IIx} + C_{IIx} C_{IIy}). \]

Defining new variables \( Y_0 \ldots Y_{14} \):

\[
Y_0 = 4 \left( \frac{1}{a^2} + \frac{|\Delta|_{11}}{2|\Delta|} \right),
\]
\[
Y_1 = \frac{1}{a^2} + \frac{|\Delta|_{12}}{2|\Delta|} - \frac{|\Delta|^2_{12}}{Y_0 |\Delta|^2},
\]
\[
Y_2 = \frac{1}{a^2} + \frac{|\Delta|_{11}}{2|\Delta|} - \frac{|\Delta|^2_{13}}{Y_0 |\Delta|^2},
\]
\[
Y_3 = \frac{|\Delta|_{12}}{|\Delta|} + \frac{2|\Delta|_{13}|\Delta|_{12}}{Y_0 |\Delta|^2},
\]
\[
Y_4 = \frac{1}{a^2} + \frac{|\Delta|_{12}}{2|\Delta|} - \frac{|\Delta|^2_{14}}{Y_0 |\Delta|^2},
\]
\[
Y_5 = -\frac{2|\Delta|_{13}|\Delta|_{12}}{Y_0 |\Delta|^2} + \frac{|\Delta|^2_{14}}{|\Delta|},
\]
\[
Y_6 = \frac{|\Delta|_{14}}{|\Delta|} - \frac{2|\Delta|_{13}|\Delta|_{13}}{Y_0 |\Delta|^2},
\]
\[
Y_7 = -\frac{2}{a^2} + \frac{4|\Delta|_{12}}{Y_0 a^2 |\Delta|},
\]
\[
Y_8 = -\frac{2}{a^2} + \frac{4|\Delta|_{13}}{Y_0 a^2 |\Delta|},
\]
\[
Y_9 = -\frac{2}{a^2} + \frac{4|\Delta|_{14}}{Y_0 a^2 |\Delta|},
\]
\[
Y_{10} = \frac{Y_0 Y_1}{4} \left( Y_2 - \frac{Y_3^2}{4 Y_1} \right),
\]
\[
Y_{11} = Y_4 - \frac{Y_3^2}{4 Y_1} - \frac{\left( Y_6 - \frac{Y_3 Y_5}{2 Y_1} \right)^2}{4 \left( Y_2 - \frac{Y_3^2}{4 Y_1} \right)}.
\]
\[ Y_{12} = -\frac{4}{a_2} \frac{4}{Y_0 a^4} + \frac{Y_7^2}{4Y_1} + \left( \frac{Y_8 - \frac{Y_7 Y_3}{2Y_1}}{4} \right)^2, \]

\[ Y_{13} = Y_9 - \frac{Y_7 Y_5}{2Y_1} - \frac{Y_6 - \frac{Y_3 Y_5}{2Y_1}}{2} \left( \frac{Y_2 - \frac{Y_3}{4Y_1}}{4} \right)^2, \]

\[ Y_{14} = Y_4 \frac{Y_3^2}{4Y_1} - \frac{Y_6 - \frac{Y_3 Y_5}{2Y_1}}{2} \left( \frac{Y_2 - \frac{Y_3}{4Y_1}}{4} \right)^2, \]

we can write the result as:

\[ \sigma_i^2 C_i(\xi, \eta) = \frac{1}{4|\Delta|^{1/2}} \left[ \frac{1}{Y_{10}(Y_{11})} \right]^{1/2} \exp \left[ \Pi_0^2 \left( Y_{12} + \frac{Y_{13}^2}{4Y_{14}} \right) \right]. \quad (27) \]

The graphs obtained for the two-dimensional case are not shown because the values obtained for the autocorrelation function of the image intensity in two dimensions are so very small when we are considering the comparison with the same geometry and parameters considered in Alvarez-Borrego [1, 10] where a circ function was used. However, this formula can be used with other geometries.

5. Conclusions

The results show that with a Gaussian glitter function it is possible to obtain an analytical relation for the correlation functions \( C_i(\tau) \) and \( C_i(\xi, \eta) \) and not only a numerical one as was found using the circ or rect functions. But upon the analysis of the behaviour of the analytical relations obtained in different geometries it was observed that the values obtained for the variance and for the correlation were quite poor. Also, the image obtained using the Gaussian glitter function left us with the initial problem of having a range of grey values in the image, which is a difficulty when it is desired to apply this model to real physical situations. We conclude that the rect or circ functions are of higher utility in the application of the model than the Gaussian glitter function.

Acknowledgment

A grant for this work was received from the National Council of Science and Technology of Mexico through project 1884P-T ‘Determinación de propiedades estadísticas de la superficie del mar a partir de imágenes remotas’.

References

Statistics of rough random surfaces


