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Tree-like branching networks and allometric laws

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Abstract. The properties of simple tree-like branching networks are used to derive global characteristics of the arrangement, such as allometric laws associated with the structure of the system. The networks are characterized by a fixed number of branches developed in each generation, and the weight or magnitude of the branches in successive generations. Conceptual examples are considered (networks of masses, electric resistors, and elastic springs) each one of them characterized by a specific recurrence relation between generations. The properties of the networks are compared with their corresponding spatial scales in order to derive allometric scaling laws. It is shown that under specific approximations of the length and cross section of the branches, some allometric exponents reported in the literature are recovered (e.g. the 3/4-law for metabolism in biology or conductivity scaling in pore networks). The formulation allows different choices of the weight parameters which enables the derivation of new power-laws not reported before (as far as the authors know). Furthermore, the results can be generalized to networks made with different elements, while obeying equivalent relations.

Keywords: Tree networks, Branching networks, Allometric laws

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1. Introduction

We study general properties of tree-like branching networks which develope their structure as sets of branches starting from a common origin (Figure 1). A fundamental aspect of the theory is to consider local and global symmetries with respect to the direction of growth (e.g. left sketches in the figure). The study is based on two main aspects. First, the physical nature of the elements that constitute the networks is considered: masses, resistors and springs. Thus, we are interested in global properties such as the total mass, resistance or elastic coefficient of the whole system, respectively. Secondly, the properties of the networks are compared with their corresponding spatial scales (length, area and volume) in order to derive both new and well-known scaling power-laws.

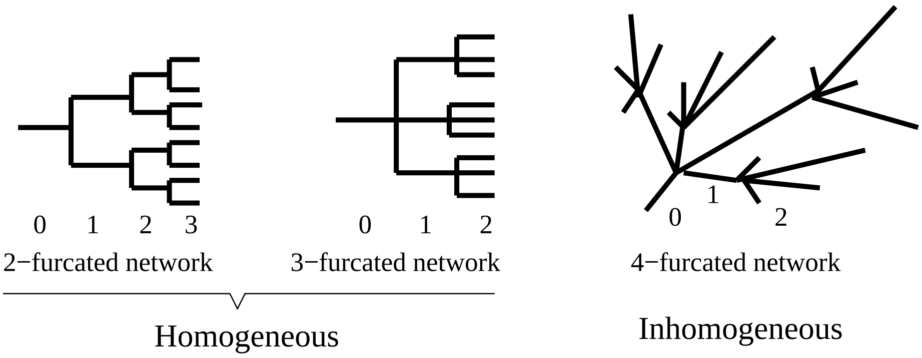

Figure 1: Schematic representation of tree-like networks with β branches in each generation.

2. Tree networks defined by a recurrence relation

We consider networks characterized by β branches developed in each generation, whose weight or magnitude is α times the magnitude in previous generation (with α a real, positive number). For instance, consider a network of resistors, where the first branch has an electric resistance r, the branches in the second generation αr, those in the third generation $\alpha^2 r$, and so on. Then we apply the additive properties of resistors in parallel and in series to obtain the total resistance of the network after n generations (which is given by a simple geometrical series). Analogous results are found for networks of masses m and springs with coefficient k, which obey different additive rules.

For example, the total equivalent mass, resistance and elastic coefficient of a network of masses, resistors and springs, respectively, are given by

$$\mu_n = \frac{1-(\alpha\beta)^{n+1}}{1-\alpha\beta}m, \qquad \rho_n = \frac{1-(\alpha\beta^{-1})^{n+1}}{1-\alpha\beta^{-1}}r, \qquad \kappa_n = \frac{1-\alpha^{-1}\beta^{-1}}{1-(\alpha^{-1}\beta^{-1})^{n+1}}k, \tag{1}$$

for $\alpha\beta \neq 1$, $\alpha\beta^{-1} \neq 1$ and $\alpha^{-1}\beta^{-1} \neq 1$, respectively. The relevance of these relations is that one can examine the properties of the system for arbitrary number of branches, weights, and for both finite and infinite arrangements.

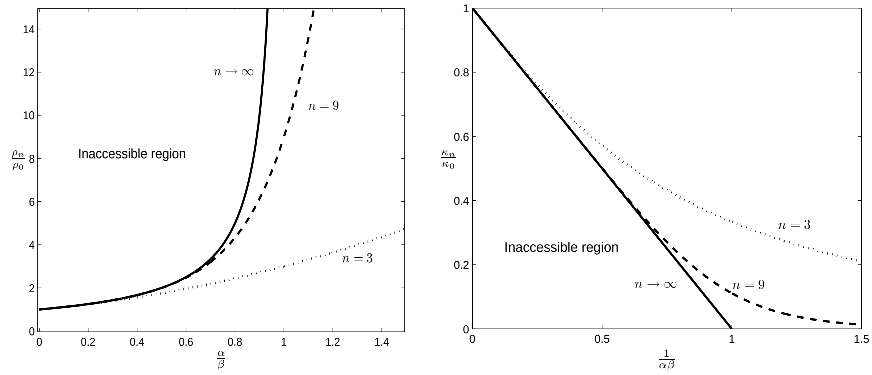

Figure 2: (a) Equivalent value of an electric network in terms of the weight and branching degree: ρ_n vs α/β. The lines indicate the equivalent value for $n \to \infty$ (solid), $n = 9$ (dashed) and $n = 3$ (dotted). (b) Same for an elastic array: κ_n vs $1/\alpha\beta$.

For instance, in Figure 2 we plot the equivalent values for resistors and springs for $n = 3$, 9 and ∞ generations. A remarkable property of the infinite electric network is that it has a finite resistance as long as the weight α is smaller than the number of branches β, that is $\alpha\beta^{-1} < 1$.

Besides the three types of idealized networks, the formulations can be applied for arbitrary networks whose elements obey equivalent recurrence relations. For instance, the relation for the network of resistors also works for coils; the recurrence formula of the elastic network can be applied for capacitors, and so on. This generality is a fundamental advantage to explore the behaviour of a great number of different systems from a unified point of view.

3. Scaling laws

The properties of the networks are compared with their corresponding spatial scales in order to derive allometric scaling laws. The formulation assumes a fractal-like structure of the branches, whose length and area scale as γ and σ, respectively, and arbitrary inhomogeneity α. Additional assumptions are area and space-filling approximations [1], as well as the preservation of an intensive property (such as mass density or resistivity) in all branches. The results are scaling laws of the normalized equivalent values of the networks ($\hat{M}$, $\hat{R}$ and $\hat{K}$, for masses, resistors and springs, respectively), in terms of their spatial magnitudes (length $\hat{L}$, area $\hat{A}$ and volume $\hat{V}$), as shown in Table 1.

Some of the obtained scaling laws have been studied before, such as the

Table 1: Scaling laws in the allometric model: $\sigma = \beta^{-1}, \gamma = \beta^{-1/3}$, α as indicated.

Property	Weight α	Length	Area	Volume
Mass	$\alpha = \gamma\sigma$	$\hat{M} = \hat{L}^4$	$\hat{M} = \hat{A}^{4/3}$	$\hat{M} = \hat{V}$
Resistance	$\alpha = \gamma/\sigma$	$\hat{R} = \hat{L}^{-2}$	$\hat{R} = \hat{A}^{2/3}$	$\hat{R} = \hat{V}^{-1/2}$
Elastic coeff.	$\alpha = 1/(\gamma\sigma)$	$\hat{K} = \hat{L}^{-4}$	$\hat{K} = \hat{A}^{-4/3}$	$\hat{K} = \hat{V}^{-1}$

electric resistance in terms of the inverse of the square root of the total volume $R \sim V^{-1/2}$ [2]. Another case is the hydraulic conductivity C, which can be estimated as the inverse of the total resistivity in a porous media. Using the analogy with the electric resistance, $C = R^{-1}$, implies that $C \sim V^{1/2}$, which is a well-known empirical law in geohydrology [3]. The present formulation also recovers some previous models in biology, such as the scaling laws reported in [4], denominated as *fractal biological*, which state that $M \sim L^4$, $M \sim A^{4/3}$ and $M \sim V$.

4. Conclusions

Tree-like networks with local and global symmetries are examined. The networks are defined by the recurrence relation of a given property between generations, which allows one to find analytical expressions for their equivalent values over the whole arrangement. When considering their spatial scales, the formulation recovers the fractal biological scaling in a very natural way without invoking efficiency or dynamical arguments, i.e. it is less restrictive than previously reported theories. In the vein of [4], the scaling laws are derived by taking advantage of their multiple symmetries. It also recovers some other power-laws for electrical and hydrological networks with no need of elaborated hierarchical trees. More importantly, establishes new power-laws which would be relevant to test, and sets the basic formulation to search for different models instead of the allometric one used here.

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