Self-oscillations of a two-dimensional shear flow with forcing and dissipation

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Two-dimensional shear flows continuously forced in the presence of dissipative effects are studied by means of numerical simulations. In contrast with most previous studies, the forcing is confined in a finite region, so the behavior of the system is characterized by the long-term evolution of the global kinetic energy. We consider regimes with $1 < \Re_1 << \Re$, where $\Re_1$ is the Reynolds number associated with an external friction (such as bottom friction in quasi-two-dimensional flows), and $\Re$ is the traditional Reynolds number associated with Laplacian viscosity. Depending on $\Re_1$, the flow may develop Kelvin-Helmholtz instabilities that exhibit either regular or irregular oscillations. The results are discussed in two parts. First, the flow is limited to develop only one vortical instability by choosing an appropriate width of the forcing band. The most relevant regime is found for $\Re_1 > 36$, in which the energy maintains a regular oscillation around a reference value. The flow configuration is an elliptical vortex tilted with respect to the forcing axis, which oscillates steadily also. Second, the flow is allowed to develop two Kelvin-Helmholtz billows and eventually more complicated structures. The regimes of the one-vortex case are observed again, except for $\Re_1 > 135$. At these values, the energy oscillates chaotically as the two vortices merge, form dipolar structures, and split again, with irregular periodicity. The self-oscillations are explained as a result of the alternate competition between forcing and dissipation, which is verified by calculating the budget terms in the energy equation. The relevance of the forcing-vs.-dissipation competition is discussed for more general flow systems. Published by AIP Publishing. https://doi.org/10.1063/1.5020130

I. INTRODUCTION

Turbulent flows may be subject to some external forcing and, simultaneously, possess a dissipative mechanism that drains the injected energy. In planetary fluid dynamics, some examples are the Earth’s atmosphere and oceanic currents in large scale basins, as well as the atmospheres of giant planets. Due to their turbulent nature, these systems undergo intense fluctuations and irregular motions in time and space. Despite this, they often exhibit persistent features, such as jets, vortices, and undulatory structures (some examples are the well-known California current system in the northeastern Pacific Ocean, the Jet Stream at the upper troposphere, the Red Great Spot on Jupiter, and the polar Saturn’s Hexagon).

Flow patterns in the presence of forcing and dissipation may be quasi-steady and exhibit some periodicity manifested as regular, observable oscillations. This was noted in the well-known rotating annulus experiments originally performed by Hide (1958), in which a differentially heated rotating flow was used as a basic model of the Earth’s atmosphere. Based on these experiments, Lorenz (1963) studied the so-called “vortices” by analyzing simple dynamical models consisting of truncated series of expanded variables. Lorenz emphasized that, in order to explain the oscillations of the system, it was essential to include forcing and dissipation in the models. An important outcome from this classical paper and numerous later studies is that the oscillations are modulated by the characteristic parameters of the problem (e.g., Pfeffer and Chiang, 1967).

Shear flows may be subject to the considerations mentioned earlier and display quasi-periodic behaviors when getting unstable (as we will show here). However, their dynamics have been mainly studied in the absence of external forces and dissipative effects or in high Reynolds number regimes. Indeed, there is abundant literature on freely evolving shear flows; lucid accounts of classical results and early studies are given by Michalke (1964) and Ho and Huerre (1984). Here we make a brief summary related to our own study.

The instability mechanism of velocity profiles with an inflexion point is inviscid, so the presence of viscosity only or delays the process (Michalke, 1964). In a planar geometry, unstable shear flows develop Kelvin-Helmholtz (KH) instabilities manifested as an array of vortices along the stream-wise direction. If this arrangement persists, the vortices might exhibit a regular oscillation in their shape and orientation. This vortex nutation was described by Zabusky and Deem (1971) for nearly parallel shear flow profiles in two dimensions. Since then, the problem has been thoroughly studied in numerous articles using different approaches [e.g., the point-vortex analysis by Aref and Siggia (1981)]. In a numerical and analytical study, Miura and Sato (1978) discussed that the vortex nutation is due to coupled oscillations between the vortex amplitude and phase. Ho and Huerre (1984) explained the vortex oscillations in terms of the orientation of their...
elliptical shape: if the vortex is negatively tilted (with respect to the cross-stream direction), the velocity components generate a positive Reynolds stress, which implies that the mean flow loses energy in favor of the disturbance; an opposite energy flux occurs when the vortex is positively tilted. Klaassen and Peltier (1985) studied the evolution of a single KH billow (as we will do in the first part of this paper) and arrived to a similar conclusion. In a more ample study, Metcalfe et al. (1987) allowed the development of subharmonic instabilities, which might lead to vortex merging and more complicated patterns.

Studies on shear flows that include both forcing and damping are less abundant. Unstable shear flows may also develop KH instabilities and, if the forcing is further increased, the flow pattern might become irregular as vortices are merged or detached [see the experiments by Niino and Misawa (1984) and the simulations by Marcus (1990)]. In the context of turbulent geophysical flows, Scott and Polvani (2007) examined the forced-dissipative problem in spherical geometry. In the review by Dolzhanskii et al. (2004, 2007) and the simulations by Marcus (1990)). In the context of turbulent geophysical flows, Scott and Polvani (2007) examined the forced-dissipative problem in spherical geometry. In the review by Dolzhanskii et al. (1990), a number of analytical and experimental systems with constant forcing and linear damping are discussed. In particular, the authors describe how the effects of linear friction modify the neutral stability curves of classical theory. Manin (1992) studied numerically the periodic merging of vortices in a shear flow and explained the oscillatory behavior of the system in terms of the interactions between different unstable modes. Obukhov (1983) described laboratory experiments on arrays of quasi-two-dimensional vortices forced by electromagnetic methods, which have been used to simulate the so-called Kolmogorov flows. As pointed out by Obukhov, a more precise description of the fluid motion requires to consider the role of an “external” friction (e.g., bottom friction) represented by a linear term in the governing equations. Batcheav (1990; 2016) performed experiments on elementary arrays of vortices and observed intrinsic, self-oscillations of the system.

In this paper, we study a two-dimensional, parallel shear flow subject to a continuous forcing and dissipative effects. Under appropriate forced-dissipative conditions, the flow develops KH instabilities and exhibits either regular or irregular oscillations, which we shall call self-oscillations (Batcheav, 1990). These consist of periodic motions of the vortical structures that arise as the shear flow becomes unstable. This behavior was observed by González Vera and Zavala Sansón (2015) by means of laboratory experiments and numerical simulations of shear flows in closed, rectangular boxes with different aspect ratios. In that study, the self-oscillations of the system were explained as a result of the alternate competition between the constant forcing applied to a shallow fluid layer and the dissipation caused by bottom friction. Our main purpose here is to solve the problem numerically using a domain that is periodic in the direction of the flow (thus avoiding the influence of closed boundaries), in order to demonstrate unambiguously the dynamical explanation of the oscillations. In addition, we provide an appropriate scaling of the governing equations and show that the oscillatory flow regimes are uniquely determined by one single parameter involving both the forcing and the dissipation.

We adopt two main approaches: First, the self-oscillations are identified by calculating the global energy of the system, as done by González Vera and Zavala Sansón (2015). In contrast with that study, we explicitly calculate the energy budget terms in order to quantify the contributions of the forcing and dissipative terms. Second, the shear flow is only allowed to develop either one or two unstable KH billows. With this restriction, we are able to fully isolate and identify the competition between forcing and dissipation.

The article is organized as follows. The formulation of the problem and a suitable scaling of the governing equations are presented in Sec. II. The results for a single KH billow are described in Sec. III, while the corresponding results for two KH structures are shown in Sec. IV. Discussions are presented in Sec. V, together with some concluding remarks.

II. FORMULATION OF THE PROBLEM

A. Governing equations and flow parameters

We study the evolution of a two-dimensional shear flow subject to the action of an external continuous forcing and linear friction. A Cartesian coordinate system $(x, y)$ is considered, where the $x$ and $y$ axes correspond, respectively, to the streamwise and crosswise directions. The motion of the fluid satisfies the Navier-Stokes equations and the continuity equation,

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + \nu \nabla^2 u + F - \lambda u,$$  

$$\nabla \cdot u = 0,$$  

where $u = (u, v)$ is the velocity field, $p$ is the pressure, $F$ is the external forcing (per unit mass), $\nu$ is the kinematic viscosity, $\lambda$ is the Rayleigh friction coefficient, and $\nabla = (\partial/\partial x, \partial/\partial y)$. The linear term is commonly used to parameterize bottom friction effects in quasi-two-dimensional models (Obukhov, 1983; Zavala Sansón and van Heijst, 2000; and Clercx et al., 2003). Unlike molecular viscosity, this term dissipates energy at the same rate for all spatial scales. The flow is confined in a rectangular domain of size $D = \{(x, y) : 0 \leq x \leq L_x, -L_y \leq y \leq L_y\}$. The boundary conditions are periodic in the $x$-direction and no-slip at $y = \pm L_y$.

The fluid is initially at rest and then set in motion by the forcing in the $x$-direction given by

$$F(y, t) = F_0 \left[ \sin \left( \frac{y}{2\delta_0} \right) e^{-\alpha y^2} \right] \left( 1 - e^{-t/\tau} \right) \hat{e}_x,$$  

where $F_0$ is the forcing amplitude, $\delta_0$ is the length scale across the shear, $\alpha$ is a parameter proportional to $\delta_0$, and $\hat{e}_x$ is a unit vector in the $x$-direction. The temporal term indicates that the forcing is initially zero and the desired profile is rapidly established at $t > \tau$, with $\tau \sim 0.001 \lambda^{-1}$. The forcing generates a shear flow that consists of two parallel opposing jets [Fig. 1(a)], similar to that of the laboratory experiments of González Vera and Zavala Sansón (2015) [see their Eq. (14) and Fig. 3(a)]. Due to the product of the sinusoidal and exponential spatial factors, the forcing is confined within a horizontal band of width $\lambda y \leq 2\alpha$. We choose $\alpha = 2.67 \delta_0$, so the width of the forcing band is about $5 \delta_0$, and the maxima are located at $y \approx 1.5 \delta_0$. The vorticity field $\omega = \partial v/\partial x - \partial u/\partial y$ is shown in Fig. 1(b). It consists of three vorticity layers with alternate signs. The central layer...
The application of the two-dimensional Fourier transform. The amplitude of the perturbation is 0.001 Kelvin-Helmholtz instability, we added a small, spatially random perturbation to the flow. The perturbation consists of a zero-mean Gaussian white noise concentrated in the shear region. The amplitude of the perturbation is 0.001, where \( u_0 \) is the characteristic velocity of the flow (defined below). This amplitude is small enough to allow the establishment of the KH instability.

The size of the domain in the \( x \)-direction, \( L_x \), and the length scale of the central shear layer, \( \delta_0 \), are set based on the stability analysis of a homogeneous mixing layer according to linear theory (Michalke, 1964). The theory predicts that, for a hyperbolic-tangent velocity profile with semi-thickness \( \delta_0 \), the dimensionless wavelength \( \ell \) of the most unstable mode is approximately 0.4449. Based on this, we study two different problems by setting \( L_x = 2\pi \). First, in Sec. III, we set the length scale of the shear layer to \( \delta_0 = 0.4449 \), so \( \ell = 2\pi = L_x \). This implies the development of only one KH billow. Second, in Sec. IV, the approximate semi-thickness is chosen as \( \delta_0 = 0.2225 \), so \( \ell = \pi = L_x/2 \), which allows the formation of two vortices. In all simulations, \( L_y = L_x \), so the total \( y \)-length of the domain \((2L_y = 4\pi)\) is large enough to neglect the effects associated with the no-slip boundaries at \( y = \pm L_y \).

Numerical simulations were performed considering different values of the forcing \( F_0 \) and the Rayleigh coefficient \( \lambda \), which are the main parameters of the problem (besides \( \delta_0 \)). In all simulations, we keep the viscosity coefficient equal to \( \nu = 1.089 \times 10^{-4} \) (in physical units), in order to ensure that dissipative effects due to linear friction are more important than Laplacian viscosity (such a difference is most quantitatively measured) with the Reynolds numbers defined in Subsection II B). Equations (1) and (2) have been solved using a two-dimensional pseudospectral method with periodic conditions described by Joly et al. (2005) and Lopez-Zazueta et al. (2016). All the variables are expressed in Fourier space by application of the two-dimensional Fourier transform. The time and spatial derivatives are computed in Fourier space, except the nonlinear terms which are evaluated in the physical space and transformed back to Fourier space. The viscous terms are integrated using a semi-implicit Crank-Nicholson scheme. The 2/3 rule is used for dealiasing. Time integration is performed by means of a third-order Runge-Kutta scheme. In each iteration, the time step \( \Delta t \) was adjusted according to the current maximum of velocity, \( u_{\text{max}} \), and gridsize, \( \Delta x \), such that Courant-Friedrichs-Lewy condition \( CFL = \frac{u_{\text{max}} \Delta t}{\Delta x} \) remains under 0.6. All calculations were computed using a grid of \( 512 \times 1025 \) collocation points equally spaced.

B. Scaling

Let be \( u_0, t_0, \) and \( l_0 \) the characteristic scales of velocity, time, and length, respectively, and \( F_0 \) the magnitude of the forcing. The velocity scale is chosen as \( u_0 = F_0/\lambda \), which reflects the two most important characteristics of the problem: the flow speed is increased (decreased) for larger forcing (damping). Since we focus our attention on time scales at which the effects of linear friction are manifested, the time scale is \( t_0 = 1/\lambda \). The length scale is \( l_0 = \delta_0 \), and the pressure scales as \( \rho u_0^2/l_0 \). Non-dimensional variables are defined as \( u' = u/u_0, t' = t/t_0, (x', y') = (x, y)/l_0, p' = p/\rho u_0^2/l_0, \) and \( F' = F/F_0, \) so the dimensionless form of the momentum equations is (after dropping out all primes)

\[
\frac{1}{Re_{l}} \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} = \frac{1}{Re_{y}} \frac{\partial^2 u}{\partial y^2} \frac{1}{F - u},
\]

where the Reynolds numbers are

\[
Re = \frac{u_0 l_0}{\nu} \equiv \frac{F_0 \delta_0}{\lambda \nu}, \quad Re_{l} = \frac{u_0}{l_0 \lambda} \equiv \frac{F_0}{\delta_0 \lambda^2}.
\]

Note that these are the usual horizontal and vertical Reynolds numbers, respectively (Clercx et al., 2003). In our case, the forcing has been included in the velocity scale, but their physical interpretation remains the same: written in terms of \( u_0 \), the Reynolds numbers \( Re \) and \( Re_{l} \) compare the time scale of the flow \( l_0/u_0 \) with the time scale of viscous effects \( (l_0^2/\nu) \) and with the time scale associated with bottom friction \( (1/\lambda) \), respectively. We will show that this is the appropriate scaling to describe the numerical results.

C. Global variables

The total energy and enstrophy of the flow in dimensionless units are defined as

\[
E(t) = \frac{1}{2} \int |u|^2 dA, \quad Z(t) = \frac{1}{2} \int \omega^2 dA,
\]

with \( dA \), an infinitesimal area element over the whole domain. Given a finite \( x \)-length of the domain, both integrals converge regardless of the \( y \)-length because the sinusoidal forcing is confined within a horizontal band (Fig. 1). This fact allows the analysis of the energy injection and dissipation in the whole system in terms of the time evolution of global integrals (González Vera and Zavala Sansón, 2015).

The evolution equation of the energy is obtained by taking the dot product of the momentum equation with the velocity,
integrating over the whole domain and applying appropriate boundary conditions [see, e.g., Clercx et al. (2003)]. This yields
\[
\frac{1}{Re_A} \frac{dE(t)}{dt} = -\frac{2}{Re} Z(t) + \frac{1}{Re_A} [W(t) - 2E(t)],
\]
where the term associated with the forcing is
\[
W(t) = \int u \cdot F dA.
\]
Equation (7) indicates that the variation of the kinetic energy is driven by the horizontal friction (proportional to the enstrophy), the rate of work done by the forcing, \( W \), and the decay due to bottom friction (proportional to the energy). In this paper, we examine flow regimes where \( 1 < Re_A << Re \), which implies that the global change of energy is mainly ruled by the competition between the forcing and the linear dissipation,
\[
\frac{dE(t)}{dt} \approx W(t) - 2E(t).
\]
We will show the conditions at which the forcing and the dissipation are balanced and the oscillatory behavior of the system when such a balance is not reached.

III. SINGLE VORTEX

We start by examining the formation of only one KH bil-low by setting the length scale across the shear \( \delta_0 = 0.4449 \). Depending on the vertical Reynolds number \( Re_A \), different flow regimes are clearly observed. We will present first some representative cases, and then we shall describe them in terms of the time evolution of the global energy.

A. Steady and oscillatory cases

For very dissipative systems, \( Re_A \leq 11.5 \), the random perturbations in the initial velocity field are fully damped. The resulting flow is steady and parallel to the forcing, as shown in Fig. 1(a). This behavior is found regardless of \( Re \). For greater \( Re_A \), the flow develops a steady, elliptical structure, as shown in Fig. 2 for a flow with \( Re_A = 27 \). During the first phase, the parallel shear flow is established by the forcing. During subsequent times, the superposed perturbation triggers the development of a Kelvin-Helmholtz instability \((t = 8.6)\) leading to the roll-up of the central vorticity layer. The flow evolves toward an elliptical core vortex connected by two vorticity filaments \((t = 10–10.8)\). Once the vortex is completely formed, its major axis is oriented against the shear induced by the forcing \((t = 13.6–16)\). The two opposite-signed vorticity layers do not develop instabilities, but they are distorted and elongated due to the action of the main vortex. As time progresses, the shape and orientation of the vortex change less, until reaching a steady state. The elliptic vortex lasts indefinitely with this configuration despite the misalignment with the forcing.

For an even higher Reynolds number, \( Re_A = 112 \), the long term behavior of the system is oscillatory (Fig. 3). As in the quasi-steady regime, the shear flow is established by the forcing and the imposed perturbation leads to the development of a Kelvin-Helmholtz instability \((t = 2–2.2)\). Afterwards \((t = 2.6)\), the central vorticity layer has started to roll up leading to the formation of an elliptic vortex connected by two vorticity filaments. In contrast to the previous case, both the shape and orientation of the vortex core change continuously. This provokes that the filaments are progressively stretched and thinned while they are being wrapped around the vortex core. Furthermore, the rotation of the vortex causes the opposite-signed vorticity layers to become strongly elongated, until they eventually break apart and generate vorticity patches \((t = 2.8)\). The system then maintains an oscillatory behavior, in which the vortex is elongated and contracted while slightly changing its orientation \((t = 3.0–3.4)\). The oscillations remain during

FIG. 2. Temporal evolution of the vorticity field (color surfaces) in a simulation with \( Re_A = 27 \) and \( Re = 4902 \). Maximum absolute vorticity values are \( |\omega_{\text{max}}| \approx 11 \). Continuous (dashed) contours correspond to negative (positive) vorticity values. The contour interval is 0.1\( \omega_{\text{max}} \). Only a fraction of the domain is shown, \((0, L_x) \times (-0.65L_y, 0.65L_y)\).
B. Flow regimes

The flow regimes are examined by calculating the nondimensional global energy (6) at different stages. Figures 4(a)–4(c) show the time evolution of $E(t)$ in several simulations representing the flow regimes. Each panel presents cases with a given $Re_A$ and different $Re$. The curves collapse, which demonstrates that the flow regimes are characterized by $Re_A$. A more detailed description is the following:

(a) Steady regime, $Re_A < 11.5$. This is a system with weak forcing or strong damping. It is characterized by a constant value of the energy, reached at a dimensional time
of about $\sim 5\lambda^{-1}$. A nearly exact balance between forcing and dissipation is established.

(b) Steady regime with damped oscillations, $11.5 < Re_{\delta} < 36$. Initially the energy increases up to a maximum at time $\sim 5\lambda^{-1}$ and then decreases to nearly $2/3$ of this value. Afterwards, the energy presents a few oscillations that are dissipated, thus reaching a stable value at long times. Although the exact occurrence of these descriptions slightly varies between different experiments, the period and the amplitude of the decaying oscillations are nearly the same. The final flow configuration is an elliptical, steady vortex (Fig. 2).

(c) Oscillatory regime, $Re_{\delta} > 36$. Now the forcing is stronger or the damping is weaker. During the initial stage, the energy reaches a maximum at about $3\lambda^{-1}$. Afterwards, the energy decreases and oscillates steadily during the rest of the simulations. The amplitude and frequency of the oscillations depend on $Re_{\delta}$, as discussed below. The flow behavior is an elliptical vortex that periodically changes its shape and orientation (Fig. 3).

The flow regimes are summarized in Fig. 4(d), which shows six cases for different $Re_{\delta}$. For larger $Re_{\delta}$, the non-dimensional energy oscillates in time with a shorter period and smaller amplitude. In addition, the mean energy is lower due to the scaling factor $F_0^2/\lambda^2 = Re_{\delta}\delta_0 F_0$, and it converges to a specific value.

Figure 5 presents the flow regimes in terms of the forcing $F_0$ and the damping coefficient $\lambda$, for different $Re_{\delta}$. In other words, we plot the quadratic $F_0$ vs. $\lambda$ curves given by

$$F_0 = Re_{\delta}\delta_0 \lambda^2. \quad (10)$$

The symbols represent the performed simulations, and the flow regimes are indicated with different line styles. These curves emphasize that the flow may behave very differently for a given set of parameters: for instance, a system with constant $\lambda$ (found along a vertical line) may present the three flow regimes by increasing the forcing. The intermediate regime is relatively narrow, and the logarithmic scale indicates that the transition between regimes is not linear.

**C. Analysis of the oscillatory regime**

Figure 6 shows the evolution of the vorticity field once the oscillations have been established in the simulation presented in Fig. 3. The plots reveal two contrasting configurations of the central vortex: its shape evolves toward an elliptical configuration with the major axis tilted against the shear of the central layer ($t = 23.6$, and at later times, the vortex is nearly axisymmetric ($t = 24.2$). The vortices repeat this process indefinitely, and we argue that the reason is associated with the competition between the continuous forcing and the linear damping.

This competition can be better understood by examining the kinetic energy budget (7). Considering $1 << Re_{\delta} << Re$, the relevant terms are the work done by the forcing $W$ and the dissipation $2E$, while the lateral viscous dissipation $2Z$ has almost no influence on the energy budget [see Eq. (9)]. Figure 7 shows the time evolution of the components of the energy growth budget, $W$ and $2E$, for the same experiment shown in Fig. 6. After the initial energy growth and decay, the oscillatory phase is clearly established at $t = 10$. The energy injection and dissipation terms oscillate around the same mean value and at the same frequency. Moreover, the amplitude of both functions is practically constant, being higher than that of $W$. The inset shows that $W$ and $2E$ are out of phase and that both quantities are balanced when the energy reaches a maximum or minimum. Note that the two times at which this balance occurs correspond with the elliptical and the axisymmetrical stages of the vortex. At $t = 23.6$, $W$ is decreasing while $2E$ reaches a maximum, and this stage corresponds with the elongated vortex (see Fig. 6). At $t = 24.4$, $W$ is increasing and $2E$ has reached a minimum, which corresponds with the quasi-circular shape of the vortex. Summarizing, the energy is maximum (minimum) when the vortex is more elliptical (circular).
These results can be represented with a simple empirical model based on Eq. (9). Since the energy oscillations are very regular, it can be assumed that \( E(t) = E_0 \sin(ft) \), with \( E_0 \) being the energy amplitude with respect to the mean energy and \( f \) being a constant frequency. The forcing term yields

\[
W(t) = E_0[f \cos(ft) + 2 \sin(ft)].
\]  
(11)

By using elementary trigonometric relations, this function can be written as

\[
W(t) = W_0 \sin(ft + \phi),
\]

(12)

where the amplitude and phase are

\[
W_0 = E_0(f^2 + 4)^{1/2}, \quad \phi = \tan^{-1}(f/2),
\]

(13)

respectively. For the oscillations shown in Fig. 7, the non-dimensional period is approximately 1.6 or \( \pi/2 \), and therefore the corresponding frequency is \( f \approx 4 \). Thus, the amplitude is \( W_0 \approx 4.27E_0 = 2.13(2E_0) \), and the phase between \( W \) and \( 2E \) is \( \phi = \tan^{-1}(2) \approx 0.35\pi \). These values are in good agreement with those calculated numerically. The dimensional period of the oscillations is \( T \approx (\pi/2) \cdot t^{-1} \).

IV. TWO VORTICES

Now we analyze the flow evolution when the shear region is thinner by setting \( \delta_0 = 0.2225 \). This value allows the formation of two KH billows when the flow becomes unstable.

A. Flow behaviour

For \( Re_\lambda < 135 \), the three flow regimes found for the one-vortex case are also observed when two vortices are allowed...
to form. For low $Re_A$, the shear does not develop any billow and the final configuration is a rectilinear flow identical to the forcing. As $Re_A$ increases, the damped oscillations are observed, but now the final configuration is two steady elliptical vortices instead of just one, as in Fig. 2. Finally, the oscillatory regime consists of two vortices that oscillate periodically, analogously to that shown in Fig. 3.

The main difference in the two-vortex simulations is that the flow behavior changes dramatically when $Re_A > 135$. An example is shown in Fig. 8 for a simulation with $Re_A = 225$ and $Re = 10\,213$. After the two vortices are formed at $t = 4$, they strongly interact and generate intense negative vorticity patches ($t = 9.2$). Later, three highly complex processes occur: (i) the vortices merge ($t = 10$), (ii) parts of them are paired with opposite-sign vorticity patches, forming dipolar structures that drift in the $\pm y$ directions ($t = 10, 12.2$), and (iii) the remaining structure within the horizontal band where the forcing is acting develops two vortices again ($t = 13$). Since the dipoles move toward regions with no forcing, they are rapidly damped. Thus, the dipoles remove energy from the central region, and then the presence of the continuous forcing re-establishes the two-vortex configuration. As a result, the process is repeated but now with an irregular and unpredictable periodicity. We verified this behavior for different vertical and horizontal Reynolds numbers.

B. Regular and chaotic oscillations

The flow regimes of the two-vortex system are analyzed with the time evolution of the global energy. Figure 9 presents the energy curves for different $Re_A$. When $Re_A < 135$, the steady and oscillatory regimes observed in the one-vortex configuration are clearly observed too. For $Re_A = 225$, the energy evolves in a chaotic fashion that is clearly unpredictable. This irregular oscillation constitutes a new regime only observed in the two-vortex system. As shown in Fig. 8, it is due to the occurrence of vortex merging processes, as well as the formation and subsequent dissipation of dipolar structures.

It is useful to compare with the results obtained for the one-vortex case. This is shown in Fig. 10, which presents the time evolution of $E(t)$ in one- and two-vortex simulations having the same $Re_A$ but different $Re$. The behavior of $E(t)$ is very similar to the corresponding case in the one-vortex system for moderate $Re_A$, as illustrated in panels (a)–(c). The curves are nearly identical in each of the three regimes but with different magnitude, being lower than that of the two-vortex case. Thus, the flow regimes in both systems are characterized by $Re_A$.

The energy evolution for $Re_A = 225$ is shown in Fig. 10(d).
The one-vortex system maintains regular oscillations, while the two-vortex case becomes chaotic.

Figure 11 presents some characteristics of the oscillations for the one- and two-vortex systems as a function of the vertical Reynolds number. In both cases, the mean energy $E_{\text{mean}}$ during the oscillations diminishes with $Re_\lambda$ [see also Figs. 4(a) and 9]. A similar behavior is observed for the amplitude, except for low $Re_\lambda$. When comparing both systems, it is noted that the mean energy and the amplitude in the two-vortex case are smaller than those for the one-vortex simulations (as shown in Fig. 10). The frequency of the oscillations grows with $Re_\lambda$ almost in a linear fashion and is nearly the same in both systems.

V. DISCUSSION AND CONCLUDING REMARKS

We studied a continuously forced shear flow with horizontal and vertical (linear) dissipation, which can be described by
the Reynolds numbers defined in (5). The flow was allowed to develop one or two KH billows (Secs. III and IV, respectively), so the appearance of more subharmonic modes is suppressed. For $1 < Re_A \ll Re$, both systems present different regimes that are determined by the vertical Reynolds number $Re_A$. The regimes are characterized by means of the long-term behavior of the total kinetic energy $E(t)$, though the same can be determined by using a different global quantity, for instance, the total enstrophy.

The flow regimes are identified in the $Re$ vs. $Re_A$ diagram presented in Fig. 12. The plot includes all the simulations with one and two vortices presented in this work. The stable regime, in which the shear flow remains always along the $x$-direction, is verified for $Re_A < 11.5$, approximately. The second regime, characterized by the emergence of a steady elliptical vortex, is observed for $11.5 < Re_A < 36$. Both regimes occur in both the one- and two-vortex systems.

For $Re_A > 36$, the flow regimes are oscillatory, consisting of very regular energy oscillations around a reference value $E_{\text{mean}}$. The flow is characterized either by one or two elliptical vortices whose shape and orientation change periodically during the oscillations. The period of these motions is $O(\lambda^{-1})$, the inverse of the linear friction coefficient. When $Re_A > 135$ in the two-vortex system, the energy oscillations become chaotic: the two vortices merge and split alternately, forming and destroying dipolar structures, in such a manner that the energy evolves over irregular and unpredictable periods.

The chaotic oscillations can be interpreted by recalling that the forcing is confined within a horizontal band along the $x$-direction. When dipolar structures form and move away from this region, they transport energy that is rapidly dissipated by linear friction, which acts over the whole domain. Afterwards, global energy is increased due to the continuous forcing over the central region, generating a new two-vortex configuration, and the process is repeated again. The problem can be extended to include additional horizontal regions with alternate forcing, so the energy can propagate in the $y$-direction. This might be used as a physical model to study the dynamics of parallel jets, as those observed in giant planets [see, e.g., Scott and Dritschel (2012)].

From a dynamical point of view, we have shown and discussed that the mechanism that rules the behavior of the shear flow is the competition between forcing and dissipation. The calculation of the energy budget terms in Eq. (7) revealed the alternate dominance between the forcing term and bottom friction effects, as illustrated in Fig. 7. This mechanism was previously discussed by González Vera and Zavala Sansón (2015) who studied a similar system in laboratory experiments and numerical simulations. In contrast with that study, here we have (i) calculated and compared the energy budget terms explicitly, (ii) performed accurate simulations with no influence of lateral walls, and (iii) controlled the number of KH billows that are allowed to develop in the streamwise direction.

The system is “non-adiabatic” in the sense that mechanical energy is continuously injected and dissipated (Lorenz, 1963). Therefore, the nutation observed in inviscid flows (reported in numerous studies cited in the Introduction) is essentially different from the self-oscillations sustained by the forcing-vs-dissipation competition. Inviscid effects are mainly associated with the interaction of different oscillating modes. This does not preclude that both mechanisms may interact together, as probably occurs in the experiments of Batchaev (1990; 2016) and in the simulations of Manin (1992). In any case, the alternate competition between forcing and dissipation must be considered when studying the dynamics of more complicated systems, such as the vacillations observed in the rotating annulus experiment. Although Lorenz (1963) did not discuss the vacillations in terms of this competition, he incorporated forcing and dissipation terms as indispensable ingredients in his theoretical model. Thus, a conceptual approach for future analyses is the existence of oscillatory or quasi-oscillatory regimes in even more complicated systems, such as oceanic or atmospheric flows, that emerge as a result of the

FIG. 12. Nondimensional parameter space ($Re$, $Re_A$) including all the simulations with one and two vortices. Symbols indicate the flow regimes explained in the text.
competition between forcing and dissipative effects, besides the combination of different unstable modes.

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