Solutions of Continental Shelf Waves Based on the Shape of the Bottom Topography*

Luis Zavala Sansón *, CICESE

Abstract
Solutions of barotropic, rigid-lid topographic waves over an infinite family of continental shelves characterized by a shape parameter are derived. The bottom topographies are defined by a depth profile proportional to $x^s$, where $x$ is the offshore coordinate and $s$ is a real, positive number. As expected, the resulting continental waves are characterized by subinertial frequencies and by their propagation along the coast with shallow water to the right (left) in the Northern (Southern) Hemisphere. The wave structure and the dispersion relation are written as a function of the parameter $s$, which essentially indicates that waves over steeper shelves have higher frequencies and phase speeds. As an example, we determine the shape parameter of the continental slope at four locations along the Eastern Pacific Ocean, and then we calculate the corresponding wave properties. Another aim of the paper is to underline the structure of the solutions in terms of the associated Laguerre polynomials, which allow the introduction of the shape parameter. This point is discussed to the light of previous studies that have reported solutions in terms of Laguerre functions.

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* Contact: lzavala@cicese.mx
1. Introduction

Coastal trapped waves in the oceans are oscillatory motions affected by the Earth's rotation and are confined within a vicinity of coastal areas and continental shelves. These waves have been thoroughly studied in the last 50 years from the theoretical, observational, numerical and experimental points of view by a large number of researchers [important reviews were written by Mysak (1980) and Brink (1991)]. This field of study is very extensive since trapped waves can be associated with several factors, such as the shape of the boundary (a straight or curved coast, an island), variable topography (the continental shelf), ambient stratification, external forcing (storms, tides), and combinations between them.

In order to develop theoretical models, past studies necessarily have done strong simplifications of the physical mechanisms involved, as we shall do here. A first consideration is the case of a homogeneous ocean, that is, in the absence of stratification. Secondly, the geometry of the coastal boundary is simplified as a straight coastline. In this barotropic system, the frequency of topographic waves can be superinertial, inertial or subinertial (see, e.g., Huthnance 1975). Superinertial oscillations, also called edge waves, are basically gravity waves affected by rotation and topography, which travel in any direction along the coast. In cases where they are not trapped within a vicinity of the boundary they are sometimes referred to as Poincaré "leaky" waves (Mysak 1968). Subinertial motions are known as shelf waves, and they are restricted to move along the coast with shallow water to the right (left) when the Coriolis parameter is positive (negative). The mechanism of these waves is associated with the conservation of potential vorticity.
In addition to their frequency and wave number, coastal oscillations are also characterized by the shape of the depth profile. Indeed, wave frequencies, and consequently phase speeds and group velocities, may vary for a given wave number over different bottom topographies. Therefore, there are neither universal solutions nor general dispersion relations for coastal trapped waves (Huthnance 1975), since they depend directly on the shape of the shelf. Thus, the bottom topography is often approximated with a well-behaved analytical function.

Considering a semi-infinite domain in which \( x \in [0, \infty] \) is the offshore coordinate, several depth profiles \( h(x) \) have been proposed. One of the earliest studies by Reid (1958) examined coastal waves over a linear profile \( h(x) = \alpha x \) where \( \alpha \) is the bottom slope. That study shows the complete set of solutions for superinertial and subinertial waves, as well as the asymptotic limits for short and long waves. An additional complication arises when considering a shelf of finite width \( L \), as shown by Mysak (1968). In that study, the wave solutions over the linear shelf were coupled with an external solution outside, where the depth was considered constant. As a result, the dispersion curves described by Reid (1958) are modified. Another well-known topography is the exponential profile \( h(x) = h_0 e^{2\lambda x} \) with \( \lambda^{-1} \) the length scale of the topography, which was used, among others, by Gill and Schumann (1974) and by Gill (1982) in the context of shelf, barotropic waves.

In this paper we examine a family of new solutions of coastal trapped waves, whose behavior is determined by the shape of the continental slope. The solutions are obtained for a bottom profile proportional to \( x^s \), where \( s \) is a positive parameter that determines the monotonic shape of the shelf. A first
Aim is to show the dependence of the wave frequency on the parameter \( s \) in order to get a better understanding of the relevance of the topographic steepness on the wave properties. The analysis is restricted to low-frequency shelf waves, although in principle, it can be extended for edge waves. This assumption means that temporal variations of the free-surface are ignored in the continuity equation, allowing the introduction of a transport function (rigid-lid approximation). In addition, the results apply for shelves with widths small compared with the external radius of deformation (Gill and Schumann 1974).

A second goal of this study is to call the attention to the offshore structure of the solutions in terms of associated Laguerre polynomials. It is shown that the analytical solutions of coastal trapped waves that include a large family of bottom topographies are given in terms of these functions. Furthermore, the solutions, characterized by a shape parameter of the shelf, are obtained by a relatively simple procedure. This was also found in a previous article for barotropic waves trapped around seamounts (Zavala Sansón 2010), which suggests that similar solutions should be found in problems with different topographic geometries.

The rest of the paper is organized as follows. The family of barotropic wave solutions is derived in Section 2, and the structure and behavior of the waves is given in detail. Section 3 includes some examples of continental shelves at four locations characterized by a shape parameter; then, some wave properties for these realistic cases are briefly discussed. In addition, we comment on the structure of the waves in terms of the associated Laguerre polynomials. The conclusions are presented in Section 4.
2. Wave solutions

2.a Derivation

Using Cartesian coordinates, the linear, shallow water equations for a homogeneous fluid layer in a rotating system are

\[ u_t - fv = -g \eta_x, \]

\[ v_t + fu = -g \eta_y, \]

\[ \eta_t + (hu)_x + (hv)_y = 0, \]

where subindices denote partial derivatives, \( u \) and \( v \) are the velocity components, \( \eta \) is the free-surface deformation, \( h \) is the fluid layer depth and \( g \) is gravity. Hereafter we consider a semi-infinite domain \((x,y)\) where \( 0 \leq x \leq \infty \) and \(-\infty \leq y \leq \infty\), bounded by a solid boundary at \( x=0 \) (the coast), and in which the fluid depth is only a function of the off-shore coordinate \( x \), that is \( h(x) \). A second consideration is the rigid-lid approximation in the continuity equation, by which we drop the time derivative of \( \eta \). This is easily understood by re-writing expression (3) in non-dimensional terms as

\[ \delta \eta_t + (hu)_x + (hv)_y = 0, \]

where \( \delta = L^2 / R^2 \), \( L \) is a horizontal length scale and \( R = (gH)^{1/2} / f^2 \) is the external radius of deformation, with \( H \) a depth scale (besides, we have made
use of $\eta \sim U/j L/g$ and $t \sim 1/f)$. The rigid-lid approximation considers wave motions with length scales much shorter than $R$, that is $\delta \ll 1$. A strong consequence of this approximation is that gravity waves are filtered out. Thus, the velocity components can be defined in terms of a transport function as

$$u = \frac{1}{h} \psi_y, \quad v = -\frac{1}{h} \psi_x.$$

(5)

The vorticity equation is derived by subtracting the derivatives of the momentum equations, which yields

$$(v_x - u_y) + f(u_x + v_y) = 0.$$  

(6)

This well-known expression states that changes of relative vorticity are associated with divergence or convergence of the flow as fluid columns experience changes of depth. This is the basic mechanism of rigid-lid, topographic waves (which can also be expressed as the conservation of potential vorticity). Substituting the divergence from the continuity equation and the corresponding expressions for the velocity components gives the following equation for the transport function

$$\psi_{xx} + \psi_{yy} - \frac{h}{h} \psi_{xt} + f \frac{h}{h} \psi_y = 0.$$  

(7)

Wave solutions are proposed of the form
\[
\psi(x, y, t) = h(x)^{1/2} \phi(x) e^{i(ky+\omega t)}
\]

which yields an equation for \( \phi \):

\[
\phi_{xx} + \left[ \frac{1}{2} \left( \frac{h_x}{h} \right)_x - \left( \frac{1}{2} \frac{h_x}{h} \right)^2 + \frac{h_x f k}{h \omega} - k^2 \right] \phi = 0 .
\]

In the context of continental shelf waves, this expression is very suitable when considering an exponential depth profile \( h \sim e^{2 \lambda x} \), where \( \lambda^{-1} \) is the length scale of the shelf, since the solutions are of the form \( \phi \sim \sin(kx) \) (see Gill, 1982, p. 410). A different topography is considered here: the depth profile is an arbitrary power of \( x \) with the following form:

\[
h(x) = h_0 (\lambda x)^s \quad \Rightarrow \quad \frac{h_x}{h} = \frac{s}{x} ,
\]

where the parameter \( s > 0 \) measures the shape of the shelf and \( h_0 \) is a depth scale. Evidently, larger \( s \) values mean steeper topographies for \( x > \lambda^{-1} \). Several studies on shelf waves use a linear profile \( s=1 \) (e.g. Reid, 1958). Figure 1 shows some examples for different values of this parameter. The advantage of this formulation is to find the wave properties as a function of the topography shape. As a result, the following expression is obtained

\[
\phi_{xx} + \left[ - \left( \frac{s}{2} + \frac{s^2}{4} \right) \frac{1}{x^2} + \frac{f k s}{\alpha x} - k^2 \right] \phi = 0 .
\]
Figure 1. Depth profiles over continental shelves of the form 
\[ h(x) = h_0(\lambda x)^s \] for \( s = 0.5, 1, 2 \) and 3. Topographic parameters are \( h_0 = 1000 \) 
m and \( \lambda^{-1} = 50 \) km.

Applying the change of variable

\[ \rho = 2kx \quad \Rightarrow \quad \chi(\rho) = \phi(x), \tag{12} \]

yields

\[ \chi_{\rho\rho} + \left[ -\left( \frac{s}{2} + \frac{s^2}{4} \right) \frac{1}{\rho^2} + \frac{fs}{2\omega\rho} - \frac{1}{4} \right] \chi = 0. \tag{13} \]
The solution is obtained in terms of the associated Laguerre polynomials with the following form (see Arfken, 1970, p. 620):

\[ \chi(\rho) = e^{\frac{\rho^2}{2}} \rho^{\frac{j+1}{2}} L_p^j(\rho) , \] (14)

where the indices \( j \) and \( p \) are defined by the following relationships

\[ \frac{j^2 - 1}{4} = \frac{s^2}{2} + \frac{s^2}{4}, \quad j > -1 \quad \in \mathbb{R} \] (15)

\[ \frac{2p + j + 1}{2} = \frac{sf}{2\omega}, \quad p \geq 0 \quad \in \mathbb{Z} \] (16)

The solutions of the first equation are \( j = \pm(s + 1) \). In general, \( j > -1 \) is a real number, and therefore there can only be solutions for arbitrary \( s > 0 \) for the positive root \( j = s + 1 \). Index \( p \geq 0 \), and the dispersion relation is derived from expression (16):

\[ \frac{\omega}{f} = \frac{s}{2p + s + 2} \] (17)

Note that all waves are subinertial over a shelf with arbitrary \( s \), and the highest frequency corresponds to the wave with \( p=0 \). Also, there is no explicit dependence of the wave frequency \( \omega \) with the wave number \( k \), in
contrast with waves over an exponential shelf (e.g. Gill, 1982). These and other properties are further described in next subsections.

In order to find the complete solutions, we note first that

$$\phi(x) = Ae^{-kx} (2kx)^{(s+2)} L_p^{s+1} (2kx), \quad (18)$$

where $A$ is an arbitrary constant with appropriate units. When substituted in (8), the full solution for the transport function is

$$\psi(x, y, t) = \psi_0 \left( \frac{\lambda}{2k} \right)^\frac{s}{2} e^{-kx} (2kx)^{s+1} L_p^{s+1} (2kx) e^{i(ky+\omega t)}, \quad (19)$$

where $\psi_0 = Ah_0^{1/2}$ is the arbitrary amplitude.

The horizontal velocity components are calculated by means of expression (5) and taking the real parts:

$$u(x, y, t) = -U_0 ke^{-kx} L_p^{s+1} (2kx) \sin(ky + \omega t), \quad (20)$$

$$v(x, y, t) = -U_0 e^{-kx} \left[ (p + s + 1) L_p' (2kx) - kx L_p^{s+1} (2kx) \right] \times \cos(ky + \omega t) \quad (21)$$

where the (arbitrary) velocity amplitude is defined as:
\[ U_0 = \left( \frac{\psi}{h_0} \right) \left( \frac{(2k)^{\frac{3}{2}}}{\lambda^{\frac{3}{2}}} \right). \]

The \( x \)-derivative of the transport function was calculated by using the following recurrence relation of the associated Laguerre polynomials (Abramowitz and Stegun, 1972):

\[ \rho \left[ L_p^{s+1}(\rho) \right] = pL_p^{s+1}(\rho) - (p + k)L_{p+1}^{s+1}(2kx), \tag{22} \]

In order to write the polynomials with indices within the range of permitted values, an additional recurrence relation was also used:

\[ L_{p-1}^{s+1}(\rho) = L_p^{s+1}(\rho) - (p + s + 1)L_p^{s}(2kx), \tag{23} \]

2.b Spatial distribution

In order to describe the wave solutions, we first show the main structure of the waves along the coast. Afterwards, we discuss the main characteristics of the propagation of the waves in terms of the parameter \( s \). For these examples, the depth scale is \( h_0 = 1000 \) m and the length scale of the topography is \( \lambda^{-1} = 50 \) km. A positive Coriolis parameter \( f = 10^{-4} \) \( s^{-1} \) is considered, which gives an inertial period of \( T \sim 0.72 \) days. Finally, recall that the wave solutions have arbitrary amplitude.

Figure 2 shows the relative vorticity and velocity fields at time \( t = 0 \) for three cases with different values of the alongshore wave number \( k \), and using
The main structure of the waves is a set of positive and negative relative vorticity patches arranged along the coast, traveling in negative $y$-direction, i.e. with shallow water to the right. The patches have maxima and minima at the coast and they rapidly decay offshore. The velocity field is composed by the corresponding gyres near the boundary. Not surprisingly, such a form is quite similar to other models of topographic waves. An important point to notice is that the waves are trapped within a distance determined by their own size along the boundary, i.e. $k^{-1}$, due to the factor $e^{-ks}$.

How is the offshore structure of the waves? Figure 3 shows the relative vorticity fields and offshore profiles for waves with $p = 1, 2, 3$ and setting $k = \lambda$ and $s = 0.5$. The vorticity profiles are taken from the positive maxima located at $(x = 0, y = 0)$ at time $t = 0$. These profiles show an oscillatory behavior with strongly decreasing amplitude for large offshore distances. Note that index $p$ indicates the number of zero crossings of the vorticity. For $p = 1$ there is one crossing: after reaching a minimum, the profile asymptotically approaches zero for large $x$. For $p = 2$ there are two zero crossings and, equivalently, for $p = 3$ there are three zero crossings. Thus, index $p$ is a natural measure of the offshore structure. Since the amplitude of the profile rapidly decreases with $x$, however, such a structure is relatively unimportant. Nevertheless, the parameter $p$ is very important for the wave propagation as shall be discussed in next subsection.
Figure 2. Alongshore structure of topographic waves over a shelf with $s = 2$, and traveling in the negative $y$-direction. Upper panels: Relative vorticity contours for waves with alongshore wave number $k = \lambda$, $2\lambda$, and $3\lambda$. In all cases $p = 1$. Thick (thin) contours indicate positive (negative) vorticity values of arbitrary magnitude. The domain is a 150 km $\times$ 300 km rectangular region. The horizontal scale $\lambda^{-1} = 50$ km is indicated by the vertical black line. Lower panels: Horizontal velocity vectors for the same waves. The magnitude of the vectors is arbitrary.
2.c Evolution of the waves

As the dispersion relation (17) indicates, the time evolution of the waves depends directly of the shape parameter $s$ and the offshore mode $p$. The dispersion relation is plotted in Figure 4 for different cases. In the first panel, the wave frequencies with different $p$ values are shown for the case of a linear sloping topography ($s = 1$). We show first this linear case as a reference since it is considered in several previous studies (Reid 1958; Mysak 1968). The plots are simple horizontal lines since there is no dependence on $k$: all waves have the same frequency regarding their alongshore size (given a fixed $p$). Dashed lines indicate the corresponding curves derived by Reid (1958), who also included long waves in his analysis (see his Figure 3). Of course, both models coincide for the short-wave regime (compared with the deformation radius), where the frequency values are given by

$$\omega = \frac{f}{2p + 3}. \quad (24)$$
Figure 3. Across structure of topographic waves over a shelf with $s = 0.5$. Upper panels: Relative vorticity surfaces for modes $p = 1$, 2 and 3. In all cases $k = \lambda$ with $\lambda^{-1} = 50$ km. The domain is a $300 \text{ km} \times 600 \text{ km}$ rectangular region. Contours as in previous figure. Lower panels: Offshore vorticity profiles of the same waves. The profiles begin at the origin and continue along the (eastward) horizontal direction. Vorticity values in the vertical axis are arbitrary.

In the work of Reid (1958) the modes are given by a positive integer $n \geq 1$, which is related with index $p \geq 0$ as $n = p + 1$, i.e. the asymptotic
value found by Reid is $\omega = f / 2n + 1$ (see his Table II showing the asymptotic limits of the dispersion relation).

Wave frequencies as a function of the shape parameter for the first 5 modes (fixed $p$) are presented in the right panel of Figure 4. These curves show one of the main points of the present study: low $s$ values imply lower frequencies, or waves with higher frequencies are developed over steep slopes. Also, the gravest mode $p = 0$ implies higher frequencies. The predicted values for linear slopes, $s = 1$, are shown with a circle (corresponding with allowed frequencies shown in the first panel). On the other hand, the star over the curve of the gravest mode $p = 0$ indicates the frequency of a wave over a topography proportional to $\chi^{1/2}$. This value was analytically calculated by Huthnance (1978) as

$$\frac{\omega}{f} = \frac{-2 + (9 + 5sk)^{1/2}}{5},$$

with $S$ a stratification parameter; for the barotropic case ($S = 0$), the present result is recovered, $\left(\frac{\omega}{f}\right) = 0.2$.

Trapped waves over this type of topography are dispersive with phase speed

$$c = \frac{\omega}{k} = \frac{fs}{k(2p + s + 2)}.$$  \hspace{1cm} (25)

Thus, larger waves (smaller $k$) travel faster. For a given wave (fixed $k$) the phase speed as a function of the shape parameter has identical behavior as the frequency curves in Figure 4 (with appropriate units). In other words, waves over steep shelves travel faster than waves over weaker slopes. In addition, over this type of topography, these waves do not transport energy along the coast, since the group velocity $d\omega/dk$ is null. This was also noticed.
in the work of Reid (1958) for $s = 1$, who pointed out that energy propagation is due to superinertial, edge waves (not present in this analysis).

![Figure 4. Left: Dispersion relation for the first 5 modes of waves over a shelf with $s = 1$ (using $f = 10^{-4}$ $\text{s}^{-1}$). Dashed lines indicate the solutions of Reid (1958). Right: Wave frequency as a function of the shape parameter. Circles indicate the frequency for the linear shelf $s = 1$ and predicted by the model of Reid (1958). The star indicates the frequency of the gravest mode over a shelf with $s = 1/2$ calculated by Huthnance (1978).]

3. Discussion

3.a Wave properties over realistic topographies

The main difference introduced in the present analysis in comparison with previous studies, is the possibility of using different bottom topographies according with the parameter $s$. It is therefore useful to examine some wave properties over realistic bottom configurations obtained from the ETOPO1 Global Relief Model (Amante and Eakins 2009).
In order to do this, we consider the continental topographies at the front of four cities along the Eastern Pacific Ocean: two at the Northern Hemisphere, Acapulco (Mexico) and Lincoln City (USA), and two at the Southern Hemisphere, Lima (Peru) and Valparaiso (Chile). The continental slope in each one of these cases is relatively uniform along the coast, i.e. it has a more or less uniform profile at least for several tens of kilometers, as shown in Figure 5. The bottom topography along the transect perpendicular to the coast is plotted in Figure 6. Note that this transect has different longitudes for each case. In addition, we plot the corresponding theoretical topography \( h(x) = h_0(\lambda x)' \) (dashed lines), as defined in (10). The topographic parameters are calculated as \( h_0 = \overline{h} \), where the bar indicates the average along the transect, and the horizontal scale \( \lambda^{-1} \) is the distance at which the fluid depth is approximately the mean depth, i.e. \( h(\lambda^{-1}) \sim \overline{h} \). We can reasonably assume the width of the shelf as \( 2\lambda^{-1} \). The shape parameter is indicated below the curves, and it is clearly different from unity (except in Acapulco). Next, we can examine the wave properties according to these values.

Table 1 shows the topographic parameters of the four topographies and the corresponding frequencies and phase speeds of the first two modes \( (p = 0, 1) \), indicated with a subindex. Recall that the frequencies for the present solutions do not depend on the wavenumber. The phase speeds are calculated for a traveling wave with wavenumber \( k = \lambda/2 \), that is, of the same order as the width of the topography. At the last column we have included the ratio \( \delta \), which is verified to be much smaller than unity in order to justify the rigid-lid approximation (when considering stratification effects this approach might not be valid). How do the calculated values compare with
previous observations at these sites? In general, the comparison is pretty satisfactory in some cases, and not so good in others, as shall be discussed below. However, it is important to recall that the present results are not intended to explain or reproduce observational results, but to provide a theory that takes into account the shape of the topography.

Figure 5. Continental slopes in front of four cities (indicated with stars) along the Eastern Pacific Ocean. (a) Lincoln City (124° W, 45° N), (b) Acapulco (100° W, 17° N), (c) Lima (77° W, 12° S), and (d) Valparaiso (71.6° W, 36° S). In all cases the contour increment is 200 m. Maximum-minimum depth contours are: 1000-0 m, 4000-0 m, 4000-0 m and 3000-0 m, respectively. Bottom topographies have 1 min resolution (Amante and Eakins 2009). The bottom profiles along the straight lines ending at the locations are shown in Figure 6.
Figure 6. Bottom topography profiles along the transects indicated in Figure 5. Solid lines: data from ETOPO1 (Amante and Eakis 2009). Dashed lines: profile $h_b(\lambda x)^s$. The topographic parameters are shown in Table 1.

For the case of the Oregon coast at Lincoln City, Table 1 shows frequencies of 0.78 and 0.54 cpd for the first two modes. These values overestimate the observations of Cutchin and Smith (1973) (0.2-0.3 cpd) performed a few kilometers south of the same location, perhaps due to the absence of bottom friction and/or stratification effects in the model. Near Acapulco, Enfield and Allen (1983) reported winter phase speeds of about 1.6-2.7 m s$^{-1}$, which are faster than the calculated for the first mode (1.0 m s$^{-1}$). The difference is expected to be larger in summer, since the waves along
the Mexican coast are mainly triggered by tropical storm forcing (see also Martínez and Allen 2004). For the Lima transect, the predicted frequencies for the first two modes are 0.21 and 0.14 cpd, which agree well with the frequency band 0.1-0.2 cpd measured by Romea and Smith (1983) who performed observations along the Peruvian coast. These authors estimate the phase speed between 2-3 m s\(^{-1}\), while the present formulation indicates 2.6 and 1.7 m s\(^{-1}\) for the first two modes, respectively (Brink (1982) obtained similar values with a more sophisticated numerical model). For the Chilean coast there are reports of traveling disturbances at 3 m s\(^{-1}\) (Pizarro et al. 1994) which is of the same order than the calculated value for the zero mode, 3.3 m s\(^{-1}\). Although the group velocity is usually more interesting, recall that it is zero in the present model.

<table>
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<th>Acapulco</th>
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Table 1. Wave properties at the four locations indicated in Figure 5.
It must be emphasized that the discrepancies observed between some of the calculated properties and available observations are likely to be explained by ambient stratification, external forcing or bottom friction effects, among other relevant mechanisms. The main point of this discussion, however, is to show that real continental slopes fit well with the profile proportional to $x^s$, and that $s$ is between 1 and 2.5, at least for the sites shown here. Furthermore, the wave properties (frequency and phase speed) are easily estimated by using a very simple formula for the dispersion relation, and in some cases realistic values are recovered.

3.b Some remarks on the use of associated Laguerre polynomial

The offshore structure of the waves studied here is given in terms of the associated Laguerre polynomials. These functions are known since the 19th century, when they were studied by the French mathematician Edmond Laguerre (born in 1834). Several applications have been found in physical problems since then. But these polynomials became particularly famous at the beginning of the 20th century, when they were used for solving the radial part of the Schrodinger wave equation applied to the hydrogen atom. The integral character of the subindex ($p$ in our case) gave rise to the quantization of the energy in this atomic model, which constituted one of the greatest achievements at the early stages of quantum mechanics (see, e.g., Arfken 1972).

In the context of topographic waves, previous models where the offshore part of the wave solutions is given in terms of Laguerre polynomials is the already cited work of Reid (1958), as well as the model reported by Mysak (1968). In both cases the bottom topography is given by a shelf with a linear
profile, $s = 1$. The solutions of Reid are obtained over a semi-infinite plane (as the one postulated here) and the offshore structure is given in terms of Laguerre polynomials $L_n$ with $n$ an integer $\geq 0$. Mysak extended this theory for a finite width shelf and the offshore solutions are Laguerre functions $L_v$ with $v$ a real number subject to some restrictions associated with the topographic discontinuity. In the present case, we found associated Laguerre polynomials (or generalized Laguerre polynomials) $L_p^p$, for waves over bottom profiles proportional to $x^s$. Thus, the point to underline here is that the associated Laguerre polynomials appear as a consequence of considering this infinite family of depth profiles, providing a more versatile, analytical model of bottom topographies. Of course, the full solution of the present waves are reduced to Reid's solutions when $s = 1$, as shown in Figure 4 (in the context of rigid-lid waves).

The use of associated Laguerre polynomials was reported before in a previous work by Zavala Sansón (2010) in the context of trapped waves around axisymmetric seamounts. In that study, rigid-lid solutions were found for seamounts with an exponential depth profile of the form $\exp(\lambda^s r^s)$, where $r$ is the radial coordinate measured from the center of the seamount and $s$ is, as in this study, a shape parameter of the topography. Large $s$ values imply a flat-topped seamount, and small $s$ means a sharp-peak mountain. The radial part of the solutions was given in terms of the associated Laguerre polynomials. Analogously to the present case, these functions give the wave structure along the direction normal to the topography contours.

Another important point related with the polynomials is the form of the wave solutions given by equation (8), which includes a factor $h^{1/2}$. Some studies proposing such a form are those of Gill (1982) for shelf waves and
Rhines (1969) for seamounts. This choice is essential to reduce the equation for the offshore structure of the wave (radial, for the case of seamounts) into a suitable form that can be solved with associated Laguerre polynomials. However, the example shown by Gill uses an exponential shelf, while the work of Rhines makes a further approximation to obtain solutions in terms of Bessel functions. Other studies (like Reid's and Mysak's) do not include the factor $h^{1/2}$, and simply express the wave solutions of the form $F(x)e^{i(kv+at)}$. It must be recalled that these latter studies, however, solve the problem including gravity waves (no rigid-lid approximation). Given the present results, we infer that the solution of the full problem (including superinertial and subinertial oscillations) over arbitrary shelves proportional to $x^s$ might be tractable with the present approach and the solutions should include associated Laguerre polynomials. An analogous situation should apply for seamounts, other topographic features or different basin geometries that can be written in a general form determined by a shape parameter $s$.

4. Conclusions

We have derived solutions of barotropic, rigid-lid topographic waves that travel over a continental shelf with depth profile proportional to $x^s$, where $s$ is a real, positive number. The analysis is restricted to the case of shelf widths much smaller than the external radius of deformation, for which gravity waves are filtered out. The remaining set of waves are usually called continental shelf waves (or sometimes quasigeostrophic waves), and they are characterized by subinertial frequencies and by their propagation along the coast with shallow water to the right (left) in the Northern (Southern) Hemisphere. The waves are trapped in the sense that they rapidly decay in
the offshore direction and move in the alongshore direction. All these properties are present in the family of solutions developed here. The novel result is that the wave structure and characteristics are derived in terms of the shape parameter $s$, so these solutions allow the description of the waves over an infinite set of continental shelves defined as powers of the offshore coordinate.

Wave frequencies as a function of parameter $s$ are given by the dispersion relation (17) and the corresponding curves were shown in Figure 4. Basically, this plot describes that frequencies are increased over steeper shelves (greater $s$ values) for all modes of oscillation. For instance, the frequency of the gravest mode ($p = 0$) over a parabolic depth profile ($s = 2$) reaches a value of $\omega = f / 2$. In contrast, over a square-root shelf ($s = 0.5$) the frequency is lower, $\omega = f / 6$.

We also showed some examples of real continental slopes for which a profile proportional to $x^5$ fits quite well. These locations are situated at the Northern (Lincoln City and Acapulco) and Southern (Lima and Valparaiso) Hemispheres. The wave properties and shape parameter were presented in Table 1. The shape parameter is within the range of 1 to 2.5. Repeating the same procedure at other locations should be a rather simple exercise and it might give some indications on the expected frequencies and phase speeds of freely evolving barotropic waves.

Another aim of this paper is to call the attention to the offshore structure of the solutions in terms of associated Laguerre polynomials. These general functions allow one to solve the topographic wave problem in the presence of depth profiles defined by a shape parameter, improving previous analytical models. Besides the present case for continental shelves, this was also shown by Zavala Sansón (2010) for barotropic waves trapped around
seamounts suitably written in terms of a shape parameter (see previous section). Thus, we strongly suggest that there might be more geometries allowing this type of structures in cases of oceanic interest, such as basins, ridges or canyons. This is part of current research by the author and collaborators.

Finally, an important remark must be made, namely, that waves over different bottom topographies might show very different characteristics. For instance, Gill (1982) considered a depth profile of the form \( h = e^{2\lambda x} \), where now \( y \) is the offshore coordinate (just rotate 90° our present coordinate system), and whose offshore solutions are proportional to \( \phi \sim \sin(ly) \) (\( l \) represents the wave number in that direction). This is essentially a different problem, since the depth field does not vanish at \( x = 0 \). In this case, the dispersion relation is

\[
\frac{\omega}{f} = \frac{2\lambda k}{k^2 + l^2 + \lambda^2}
\]  

(26)

with \( k \) the alongshore wave number (a similar expression was reported by Allen (1975) for coastal waves in a stratified ocean). These waves certainly have a different behavior with respect to those presented here. For instance, the group velocity is different from zero except at the highest frequency, where the slope of the dispersion curve changes sign. Summarizing, the behavior of topographic waves might strongly differ depending on the model considered. Or, as pointed out by other authors (Huthnance 1975): there is no universal dispersion relation for waves over arbitrary topography.
References


