The two-dimensional character of spin-up in a rectangular container

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This study presents two-dimensional (2D) numerical simulations of the spin-up of a homogeneous fluid in a rectangular container, compared with the corresponding three-dimensional (3D) simulation presented by other authors. A 2D simulation is based on a recently derived model [Zavala Sansón and van Heijst, J. Fluid Mech. **412**, 75 (2000)] including nonlinear Ekman friction effects, in addition to the well-known linear damping term. It is found that the inclusion of nonlinear Ekman terms provides a remarkable agreement between 2D and 3D simulations for times comparable with the Ekman timescale, which implies that this formulation is able to capture the essential viscous features of the 3D situation. In contrast, a simulation based on a 2D model with conventional linear friction is not able to represent the 3D case correctly. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1568749]

The spin-up from rest of a homogeneous fluid in a rectangular container is examined by comparing 2D and 3D numerical simulations. The 2D simulations are based on the model derived by Zavala Sansón and van Heijst,[1] which includes linear and nonlinear Ekman effects due to the bottom boundary layer. The 3D simulation is taken from van de Konijnenberg,[2,3] who studied experimentally and numerically the spin-up in a rectangular tank with low angular velocity. The purpose of this comparison is to show that a relatively simple 2D formulation is able to reproduce the main results obtained with an independent 3D simulation for this flow regime, during periods of time comparable to the Ekman timescale (which is usually much longer than the rotation period of the system). Furthermore, it is shown that using only the conventional linear damping effect in a 2D simulation, the 3D situation is not satisfactorily reproduced. The importance of using a 2D formulation which captures a 3D situation is that: (i) a more simple physical explanation of the observed phenomena is achieved, and (ii) computationally expensive 3D simulations can be avoided.

Essentially, the adjustment of an initially-at-rest fluid to a given angular rotation is a 3D process. The spin-up process is mainly due to viscous effects associated with the boundary layers at the lateral walls and with the Ekman boundary layer at the solid bottom. Since these layers are very thin compared with the horizontal and vertical dimensions of the container, their effects on the flow evolution can be included in 2D formulations. Since the work of van Heijst and collaborators,[4,5] the spin-up in rectangular containers has been studied by a number of authors through the last decade.[2,3,6–8] The stages of the adjustment process are well-known: (i) During the first few seconds, a flow with uniform vorticity -2Ω (with Ω the angular rotation of the tank) is established. (ii) The opposite-signed vorticity (i.e., cyclonic) produced at the lateral walls due to the no-slip boundary condition grows and separates, producing cyclonic vortices near the corners of the container. (iii) A pattern of nearly circular vortices is formed along the long side of the tank, with alternate circulations. The vortices gradually decay due to bottom friction effects, until the fluid reaches the state of solid-body rotation.

The number of vortices along the tank at the latter stage mainly depends on the aspect ratio $\delta = a/b$, where a (b) is the long (short) side of the container.[5] For instance, for $2<\delta<3$ the number of vortices is 3. The arrangement of these structures depends also on the Reynolds number defined as $\mathrm{Re} = b^2\Omega/\nu$, where ν is the kinematical viscosity of the fluid. For the three-cell pattern, small Re implies a cyclone-anticyclone-cyclone pattern (denoted by Henderson[7] as $+-+$). For higher Re the two peripheral cyclones might merge at the center of the tank, and form a $-+-$ pattern. The simulations in this paper are restricted to the low Re regime. An additional nondimensional parameter involving the vertical length scale H is the Ekman number defined as $E = \nu/\Omega H^2$.

Most of the references cited above have used 2D numerical simulations with no bottom friction effects, which show a qualitative agreement with experimental observations. As far as the author knows, the only 2D simulations including bottom friction effects were recently performed by Suh and Choi.[9] Here, the 2D model proposed by Zavala Sansón and van Heijst[1] is used:

$$\frac{\partial \omega}{\partial t} + J(\omega,\psi) - \frac{1}{2}E^{1/2}\nabla\psi\cdot\nabla\omega = \nu\nabla^2\omega - \frac{1}{2}E^{1/2}\omega(\omega+f), \tag{1}$$

where t is the time, $f = 2\Omega$ is the Coriolis parameter, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian operator, J is the Jacobian operator, and $\omega = \partial v/\partial x - \partial u/\partial y$ is the z component of the relative vorticity, with (u,v) the velocity components in the (x,y) directions, respectively. As shown by these authors, the terms proportional to $E^{1/2}$, which is much smaller than unity, represent bottom friction effects. The (nonlinear) Ekman terms in the left-hand-side of Eq. (1) are the corrections to the advective terms $J(\omega,\psi)$ due to bottom friction. The Ekman terms at the right-hand-side represent stretching effects associated with the Ekman pumping. Conventional formulations only contain the linear term. An important feature of this model is that it was derived by using the linear Ekman law, but retaining nonlinear Ekman terms in the vorticity evolution equation. The stream function ψ is related to the relative vorticity through the Poisson equation

$$\omega = -\nabla^2 \psi, \tag{2}$$

and the horizontal velocities are

$$u = \frac{\partial \psi}{\partial y} - \frac{1}{2} E^{1/2} \frac{\partial \psi}{\partial x}, \tag{3}$$

$$v = -\frac{\partial \psi}{\partial x} - \frac{1}{2} E^{1/2} \frac{\partial \psi}{\partial y}. \tag{4}$$

Written in a suitable $\omega-\psi$ formulation, the model is solved by means of a finite differences code[1] within a domain discretized by 129×129 grid points, and a constant time step of 0.1 s. These numerical parameters give a reasonably good resolution for the present case, and they allow a computationally inexpensive simulation. It is worth to emphasize that this physical model can be applied for nonaxisymmetric flows and domains, as shown in other studies,[10,11] and not only to axisymmetric cases.

The spin-up problem considered here was studied by van de Konijnenberg[2,3] by means of laboratory experiments and 3D numerical simulations. Those studies examined the spin-up of homogeneous fluids in several situations. For the present purposes, attention is focused on the spin-up in a rectangular container with low angular velocity.[3] The horizontal dimensions of the rectangular tank were $a=88.9$ cm and $b=38.9$ cm, while the fluid depth was $H=35$ cm. The angular velocity of the tank was $\Omega=0.035\ \mathrm{s}^{-1}$. For such a value the parabolic deformation η of the free surface can be ignored, since $\eta\sim\Omega^2 a^2/2g\sim0.005$ cm, which is much smaller than H. Using these values, the nondimensional numbers are $\delta=2.28$, Re=5296, $E=2.3\times10^{-4}$. The Ekman period is defined as $T_E=H/(\nu\Omega)^{1/2}=1871$ s. The 3D simulations of van de Konijnenberg were performed by using a finite volume method described by Andersson, Billdal and van Heijst.[12] They used no-slip lateral and bottom boundaries, and a flat, free surface.

A first glance of the spin-up in this container is shown by means of vorticity contours calculated with the 2D model in Fig. 1. As mentioned above, at $t=0$ s the initial vorticity distribution is uniform all over the domain, except at the sidewalls. Afterwards, boundary layer separation is observed at $t=120$–180 s, as well as the formation of cyclonic cells at the corners of the tank. The two cyclones at the end of the long sides of the container are stronger, and eventually they grow in size comparable with the anticyclonic cell at the center of the tank ($t=360$ s). For later times, a three-cell array has been formed, which slowly decays due to bottom friction ($t=960$–1440 s). These plots show that the 2D model reproduces quite well the general flow pattern calculated with the 3D simulations of van de Konijnenberg[3] (see their Fig. 3).

A stronger comparison between 2D and 3D simulations is obtained by means of scatter plots, which show the relationship between vorticity and streamfunction over the whole domain, evaluated at the flat, free surface. Figure 2(a) shows the scatter plots at six different times calculated from the 3D numerical simulation of van de Konijnenberg.[2] These plots can be compared with the corresponding 2D numerical simulation based in Eq. (1), shown in Fig. 2(b) (note that the streamfunction has been normalized with the half-length of the short side of the tank $B=b/2$). The initial flow ($t=0$ s)

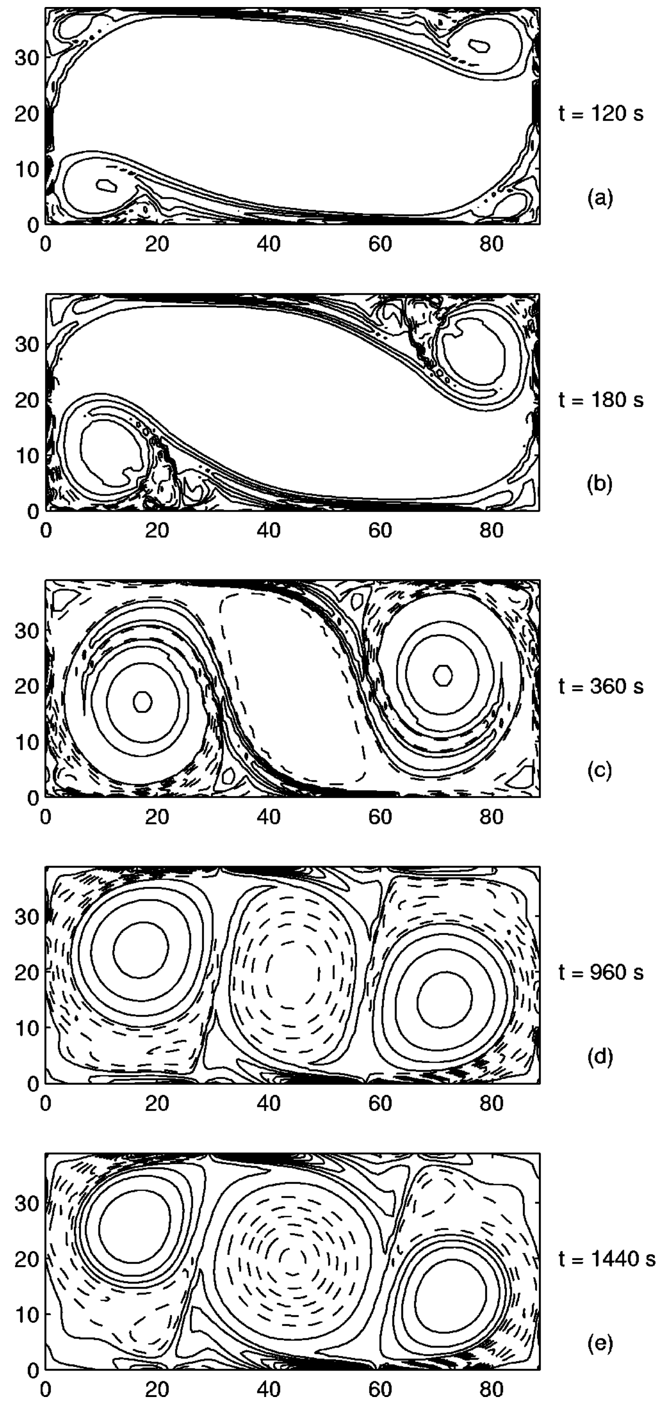

FIG. 1. Vorticity contours calculated from a numerical simulation based on the 2D model, at five different times. The rotation period and the Ekman timescale are $T=180$ s and $T_E=1871$ s, respectively. The depth is $H=35$ cm. The contour interval is chosen as $\Delta\omega=(maxv-minv)/30$, where *maxv* and *minv* are the maximum and minumum vorticity values, respectively, at a given time. Solid (dashed) lines are positive (negative) contours.

has a uniform vorticity -2Ω in both cases. At $t=120$ s and $t=360$ s boundary layer separation is produced and the cyclonic vortices are formed at the opposite corners of the tank. The dots with negative vorticity and to the left represent the fluid at the central parts of the tank, while dots with positive vorticity and to the right are those regions where the cyclonic vortices are created. Dots with ψ around zero are between cyclonic and anticyclonic regions, and dots with exactly $\psi=0$ correspond with the sidewalls. In order to facilitate the comparison with the 3D case, the results have been plotted by using the same scale as that used by van de Konijnenberg.[2] Furthermore, scatter plots of a 2D simulation without including nonlinear friction terms is shown in Fig. 2(c).

Notice the general agreement between the scatter plots of the 3D and 2D (including nonlinear terms) simulations [Figs. 2(a), 2(b)]. This can be appreciated when considering the shape of the negative and positive vorticity branches, corresponding with the central anticyclone and the two lateral cyclones, respectively. Although there are some differences in the peak numerical values (up to 25% at $t=180$ s) the scattering in both plots is clearly comparable. For later times ($t=960-1440$ s), this agreement is still noticeable, which is a remarkable result since these times correspond with almost an Ekman period ($t/T_E=0.51-0.77$). An important feature of the 2D model is that it is able to account for the faster decay of cyclonic regions compared with the decay of anticyclonic cells.[11] See for instance that the cyclonic branch at $t=960$ s has already decayed more than two times with respect to previous panel at $t=360$ s, while the anticy-

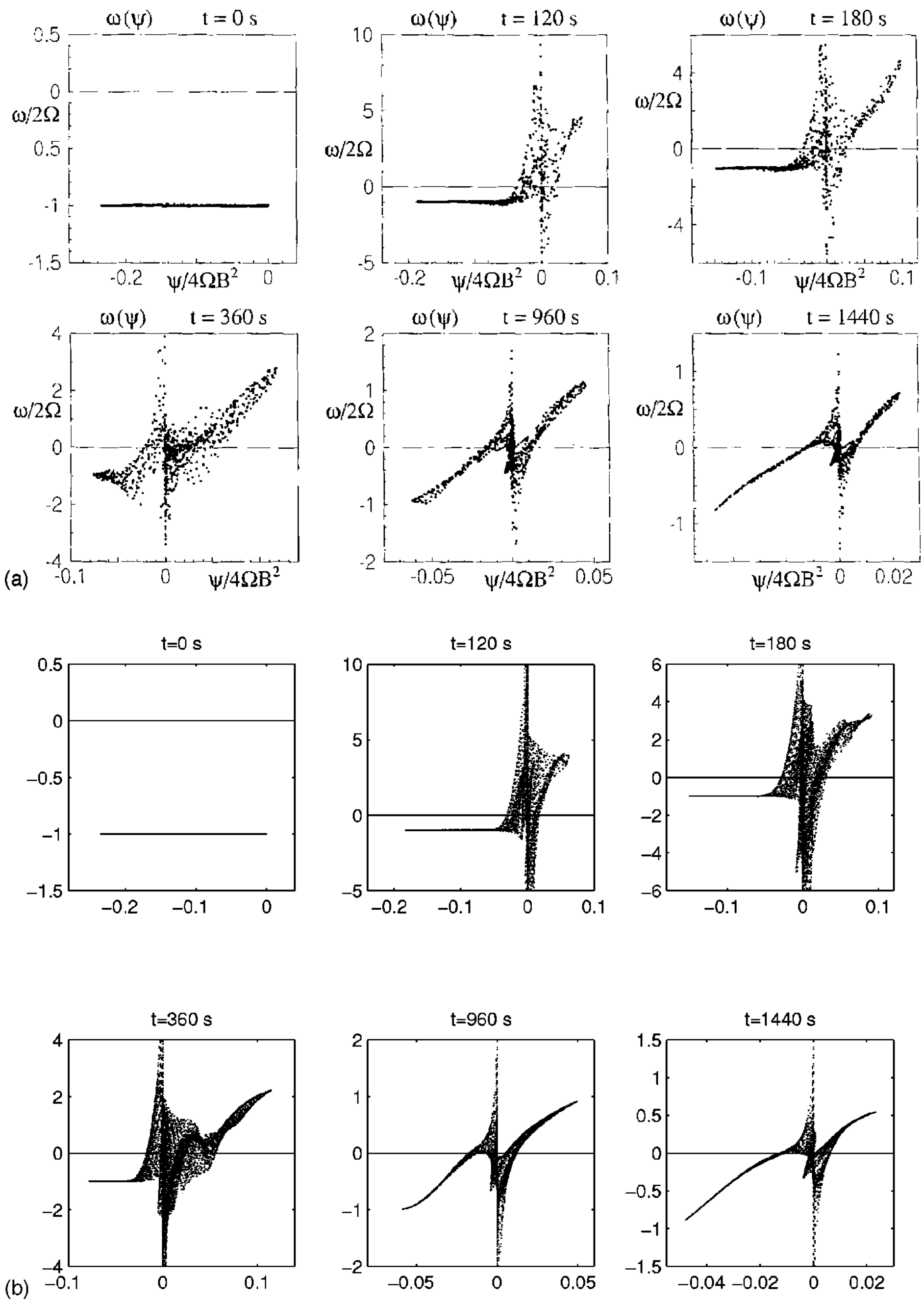

FIG. 2. Numerically calculated scatter plots showing the $\omega-\psi$ relation representing the spin-up evolution at six different times. (a) 3D simulation taken from van de Konijnenberg (1995). (b) 2D simulation based on Eqs. (1)–(2), and using the same flow parameters as the 3D situation. (c) 2D simulation with only the linear friction term (i.e., without nonlinear Ekman terms). All values in the 2D simulations are normalized in order to compare with the 3D case.

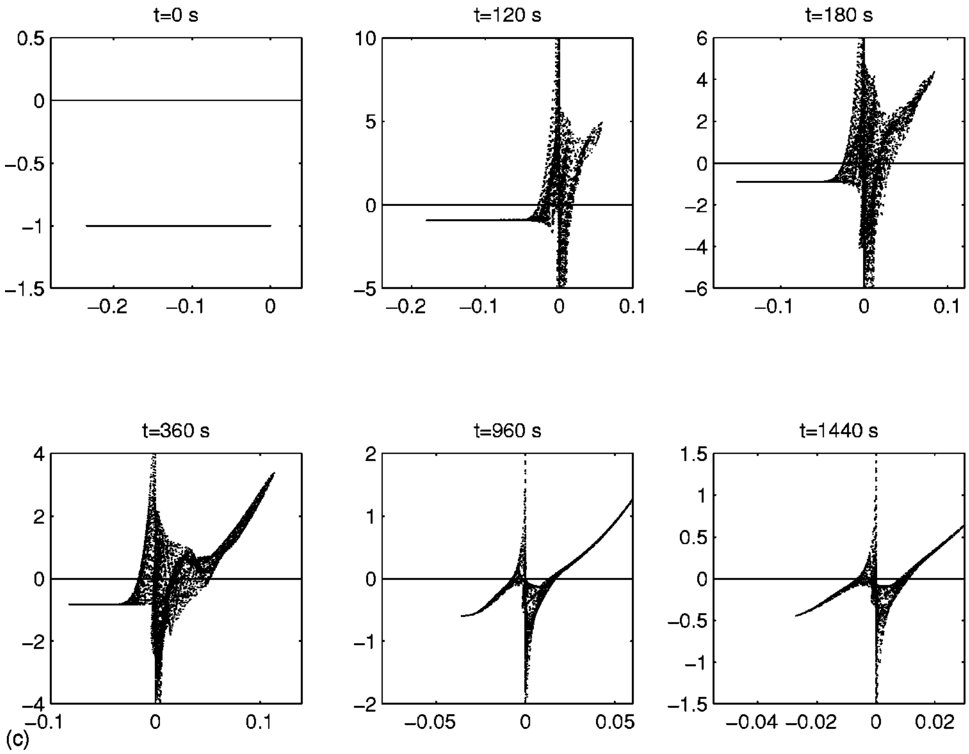

FIG. 2. (*Continued.*)

clonic branch decays much slower (see also $t=1440$ s). This dynamical behavior is not captured by the linear friction simulation [Fig. 2(c)], where the positive vorticity branch (peripheral cyclones) is still larger than the negative branch (central anticyclone) at $t=1440$ s, in contrast with the nonlinear case. There are some even finer details in the 3D example that are successfully reproduced by the 2D nonlinear simulation, such as the small branch of negative vorticity clearly observed at $t=1440$ s. These dots correspond with regions between the cyclonic vortices and the nearest lateral walls, where very weak anticyclonic vorticity is still developed. Again, the 2D linear friction case is not able to reproduce this behavior.

This study shows that the 2D formulation (1) is able to capture the essential features of a 3D numerical simulation of the spin-up in a nonaxisymmetric container for times comparable with the Ekman period. The comparison between 2D and 3D cases has been appreciated by using scatter plots, instead of using only vorticity contours or velocity arrows. The advantage of using scatter plots is that they allow a direct comparison of the values of cyclonic and anticyclonic regions all over the domain, revealing subtle details such as the faster decay of cyclonic fluid elements or the emergence of very weak structures.

It must be stressed here that the present results are limited to show that the essential physics in a particular 3D situation can be recovered with a relatively simple 2D formulation with nonlinear Ekman effects (which is also better than the conventional 2D model with only linear friction). Future work should be focused on determining the limitations of the 2D model for different Reynolds and Ekman numbers, as well as in providing a more strict quantitative comparison between scatter plots. As a final consideration, it can be said that, for the present flow regimes, nonlinear Ekman effects are necessary (in addition to the conventional linear term) and sufficient to reproduce a 3D situation. Recently, Suh and Choi,[9] performed 2D simulations of a similar spin-up problem with a physical model equivalent to Eq. (1). In that study, velocity fields calculated with such simulations as well as the time evolution of the global kinetic energy were compared with experimental measurements in the laboratory, showing a good agreement for about $0.34T_E$.

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