Quantitative estimation of effective resolution of cubed-sphere grid used in spectral element dynamical core

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KIAPS (Korea Institute of Atmospheric Prediction Systems, established in 2011) has been developing dynamical core based on the spectral element method implemented on a globally isotropic cubed-sphere grid.

The formal horizontal resolution of the model may be different from its actual (effective) horizontal resolution that can be more relevant to both dynamics and physics issues in global atmospheric modeling.

We have developed a quantitative way to estimate effective horizontal resolution of the cubed-sphere grid of the spectral element dynamical core based on spherical harmonics analysis and synthesis on the cubed-sphere grid.
Formal horizontal resolution

- **Convention**
  - Formal horizontal resolutions of the cubed-sphere grids used in the spectral element dynamical core are often denoted by ne16np4, ne30np4, ne60np4, ne120np4 and so on

- **Example**
  - 16 in ne16np4 means 16 elements along each side of cube face
  - 4 faces cover the globe along the equator
  - There are 64 (=16×4) elements along the equator
  - Each element is divided by 3×3 grid spacings for the use of 3rd order Lagrange polynomial
  - There are 192 (=64×3) grid spacings along the equator
  - ne16np4 indicates approximately 1.875° (= 360°/192)
  - ne30np4 ≈ 1°, ne60np4 ≈ 0.5°, ne120np4 ≈ 0.25°

- **Effective resolution is expected to be coarser than formal one**
  - Mapping of cube onto the sphere is not globally smooth
  - Metric tensor is not global
For any node points on sphere, SPH analysis/synthesis can be carried out in a least-square way (Fornberg and Martel 2014)

For the node data $f_i(x)$ given by $\sum_{i=1}^{N} \lambda_i \psi_i(x)$, where $\psi_i$ is the $i$-th SPH basis function

- $N = (m_{\text{max}} + 1)^2$ for triangular truncation

$$ P \lambda = \begin{bmatrix} \psi_1(x_1) & \psi_2(x_1) & \cdots & \psi_N(x_1) \\ \psi_1(x_2) & \psi_2(x_2) & \cdots & \psi_N(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(x_N) & \psi_2(x_N) & \cdots & \psi_N(x_N) \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_N \end{bmatrix} = f $$

- $\psi_1 = Y_0^0$, $\psi_2 = s Y_1^1$, $\psi_3 = c Y_1^0$, $\psi_4 = c Y_1^1$, ...

In practice, SPH analysis should be first obtained, and it requires the inversion of $P$ matrix
Inversion of the $P$ matrix
- $P$ is a $M \times M$ matrix ($M$: The degree of freedom)
- $P$'s inversion rarely exists (Fornberg and Martel 2014)
- $P$'s condition number (maximum singular value to minimum) is usually so huge that the linear system may not be numerically solved in a stable way

Pseudo inverse after truncation
- Spectral truncation of $P$ is required for SPH transform
- Truncation reduces $P_{M \times M}$ to $P_{M \times N}$ [$N$ is the number of spectral coefficients given by $\left( m_{max} + 1 \right)^2$ for triangular truncation]
- Inversion of non-square $P_{M \times N}$ using SVD

\[
P_{M \times N} = U_{M \times N} W_{N \times N} V_{N \times N}^T
\]
\[
P_{N \times M}^{-1} = V_{N \times N} W_{N \times N}^{-1} U_{N \times M}^T
\]

where $U$ and $V$ are orthogonal matrices
Singular value analysis for selected truncations

- Singular value analysis is carried out for the full matrix and 6 truncated matrices for ne16np4
- Truncations are chosen based on grid spacing along equator
  * $\Delta_h$ is grid spacing for ne16np4 (= 1.875°)
  * T75 was chosen because of stair-step distribution of condition numbers as a function of truncation (will be shown later)

<table>
<thead>
<tr>
<th>Truncation</th>
<th>Minimum zonal wavelength</th>
</tr>
</thead>
<tbody>
<tr>
<td>T96</td>
<td>3.750° (2.0$\Delta_h$)</td>
</tr>
<tr>
<td>T75</td>
<td>4.800° (2.5$\Delta_h$)</td>
</tr>
<tr>
<td>T64</td>
<td>5.625° (3.0$\Delta_h$)</td>
</tr>
<tr>
<td>T48</td>
<td>7.500° (4.0$\Delta_h$)</td>
</tr>
<tr>
<td>T32</td>
<td>11.25° (6.0$\Delta_h$)</td>
</tr>
<tr>
<td>T24</td>
<td>15.00° (8.0$\Delta_h$)</td>
</tr>
</tbody>
</table>
Results of the singular value analysis

Singular values

- SVD is computed with ScaLAPACK (Blackford et al. 1997) since the size of matrix is as huge as $10^4 \times 10^4$ for ne16np4.
- Minimum singular value for full matrix is so tiny that condition number is $O(10^{18})$.
- Condition number for T96 is also huge $O(10^{10})$.
- Condition numbers for T24, T32, T48 and T64 are all $O(1)$ so the computation of their pseudo inverse is expected to be reliable.
SPH reconstruction and relative errors

- **SPH reconstruction**

  \[ \psi_{\text{reconstruct}} = P \left( P^{-1} \left( \psi_{\text{analytic}} \right) \right) \]

  where \( \psi_{\text{analytic}} \) is an analytic real spherical harmonic function with particular zonal wavenumber \( m \) and meridional index \( l \) for triangular truncation

- **Relative errors**

  \[ E_{1,2,\infty} = \frac{||\psi_{\text{analytic}} - \psi_{\text{reconstruct}}||_{1,2,\infty}}{||\psi_{\text{analytic}}||_{1,2,\infty}} \]

  - \( L_1 \)-norm: \( ||A||_1 = \sum_i |A_i| \)
  - \( L_2 \)-norm: \( ||A||_2 = \sqrt{\sum_i |A_i|^2} \)
  - \( L_\infty \)-norm: \( ||A||_\infty = \max_i |A_i| \)
Relative errors at T96 for ne16np4

- Relative errors for reconstructed SPHs at T96
  - All the errors are $O(10^5)$ which is much larger than $O(10^{-14})$ (truncation error for double-precision variables)
  - Error structure in $m - l$ domain is substantially different from that in cases of T24, T32, T48, and T64 (see next slide)
Relative errors for reconstructed SPHs at T48
- $E_1$ and $E_2$ are $\mathcal{O}(10^{-14})$
- $E_{\infty}$'s maximum is $\sim 1 \times 10^{-13}$, but mostly $\mathcal{O}(10^{-14})$
- Errors of $\mathcal{O}(10^{-14})$ implies that errors are mostly due to the truncation of double-precision variables
- Error structure in $m - l$ domain is similar to each other at T24, T32, T48 and T64, in contrast to T96
Condition number distribution

- Stair-step distribution of condition numbers
- Condition number increases monotonically with the truncation numbers
- At some particular truncations, the rate of increase of condition number approaches zero, but condition number increases again as the truncation number increases further
- Interesting feature, but reason is not clear yet
- Effective resolution seems to be found near T45
Errors’ mean and max distribution

Mean values of $E_1$, $E_2$ and $E_\infty$
- They remain at $\mathcal{O}(10^{-14})$
- Mean error values are not good measures to find effective resolution

Max values of $E_1$, $E_2$ and $E_\infty$
- Max values of $E_1$ and $E_2$ also remain at $\mathcal{O}(10^{-14})$, and they are not good measures either
- But $E_\infty$ has clear growing trend with truncation, which is a good measure to find effective resolution
- $E_\infty$ is $\mathcal{O}(10^{-14})$ at $T57$, $T49$, $T47$, $T46$, $T45$, $T44$, $T41$, $T40$, and so on
Truncation < T49
- At truncations below T49, $E_\infty$ of $O(10^{-14})$ starts to occur frequently for ne16np4
- Effective resolution may be T45-T47

Grid imprinting arises when $E_\infty > 10^{-13}$
- No grid imprinting at T57 where $E_\infty = O(10^{-14})$
- Easy to find grid imprinting at T64
- Even for small truncation number, grid imprinting can arise (e.g., $E_\infty > 10^{-13}$ at T38)

Some examples of reconstructed SPHs ($^c Y_l^m$ or $^s Y_l^m$) with maximum $E_\infty$ in the $m - l$ spectral triangle
- $^c Y_1^0$ for T57
- $^s Y_2^2$ for T64
- $^s Y_0^0$ for T64
- $^c Y_1^0$ for T38
Reconstructed SPH and grid imprinting - II

\( \ell Y (l = 1, m = 0) \) for T57

![Map of \( \ell Y (l = 1, m = 0) \) for T57](image)

**Effective resolution of cubed-sphere grid**
Reconstructed SPH and grid imprinting - III

$^4Y (l = 2, m = 2)$ for T64

Lat (deg) vs Long (deg)

Linf Relative Error = 2.175e-13

Effective resolution of cubed-sphere grid
Reconstructed SPH and grid imprinting - IV

$\psi Y (l = 0, m = 0)$ for T64

(a) Constant field - value is 0.2821

(b) Linf Relative Error = 1.230e-13
Reconstructed SPH and grid imprinting - V

$^cY(\ell = 1, m = 0)$ for T38

**Figure a**
- Color map of $^cY(\ell = 1, m = 0)$ for T38 grid. The color scale ranges from -0.5 to 0.5.

**Figure b**
- Close-up view of the high-resolution details, indicating very small values with a Linf Relative Error of $1.038 \times 10^{-13}$. The color scale here is more fine-grained, capturing the intricate patterns of the grid imprinting.
Algorithm to compute SPHs on scattered node sets on a 2-sphere has been developed.

From SPH-reconstruction tests,
- It is found that $E_1$ and $E_2$ are not good measures to find effective resolution since their mean and maximum always remain at $O(10^{-14})$.
- $E_\infty$’s maximum is found to grow rapidly with truncation. Above T60, $E_\infty$’s maximum is $O(10^{-13})$, larger than double-precision truncation error.

Effective resolution may be determined by $E_\infty$’s maximum (i.e., whether or not grid imprinting is severe).
- Large $E_\infty$ maximum is related to grid imprinting.
  - T57 is the max truncation where grid imprinting is not found.
  - Truncation around T45–T47 seem to represent stably SPHs with almost no grid imprinting.
Discussion

- Same analysis for the higher formal resolutions
  - DOF for ne16np4 = 13826 \([\mathcal{O}(10^4)]\)
  - DOF for ne30np4 = 48602 \([\mathcal{O}(10^4)]\)
  - DOF for ne60np4 = 194402 \([\mathcal{O}(10^5)]\)
  - DOF for ne120np4 = 777602 \([\mathcal{O}(10^5)]\)

- Issues in SVD as formal resolution increases
  - For ne30np4, no problem at most of truncations, but SVD routine in ScaLAPACK gives negative singular values as the SPH matrix is less truncated
  - For ne60np4, ScaLAPACK SVD routine produces negative singular values at most of truncations except for some severely truncated cases
  - This SVD issue seems to be due to the fact that the size of matrix is too large
  - It is possible to be unable to compute effective resolutions directly for the formal resolutions higher than ne60np4
Thanks

Questions?