The "hydrostatic paradox," apparently first recognized by the Flemish scientist Simon Stevin (1548–1620), is the realization that the force due to fluid pressure on the bottom of a vessel can be considerably greater or considerably less than the weight of fluid in the vessel. If the sides of the vessel slope generally inward from bottom to top, the force on the bottom is greater than the weight of fluid contained. If the sides slope generally outward from bottom to top, this force is less than the weight of fluid. These apparently paradoxical observations can be explained by noting that the necessarily perpendicular forces exerted on the sloping walls by the fluid have vertical components that can push up or push down, depending on which way the walls slope.

An instructive example for showing the hydrostatic paradox and its quantitative resolution is a hollow right circular cone resting on its base as shown in Fig. 1. The weight of fluid of uniform mass density \( \rho \) required to fill such a cone-shaped vessel of base radius \( R \) and height \( h \) is

\[
wt = \frac{1}{3} \pi R^2 h \rho g
\]

where \( g \) is the gravitational field strength. The downward force on the base of the cone due to fluid pressure \( (F = PA, P = \rho gh) \) is

\[
F_D = \pi R^2 \rho g
\]

which is triple the weight of fluid in the cone as given by Eq. (1). This is an example of the historic hydrostatic paradox.

The paradox can be resolved by showing that when all the vertical fluid pressure forces acting on the inside of the cone are taken into account, their vector sum does in fact equal the weight of fluid given by Eq. (1) and represented by the vector pointing vertically downward in Fig. 1.

The primary forces due to the varying fluid pressures acting on the inner surfaces of the cone are everywhere perpendicular to the surfaces, as represented in Fig. 2. The horizontal components of these forces sum to zero, but the yet-to-be-determined sum of the vertical components should be the weight of fluid.

Since the downward vertical force due to fluid pressure is already given by Eq. (2), it only remains to determine the net upward vertical force acting on the cone, and then to subtract it from that given by Eq. (2). The setup for this determination is shown in Fig. 3 where the \( y \)-axis is taken to be positive down-
ward and a double summation is used: the first over the azimuthal angle $\phi$ from 0 to $2\pi$, and the second over $x$ from 0 to $R$. For this setup the incremental upward vertical force component, $dF_\uparrow$, at a particular level in the fluid is given by

$$dF_\uparrow = x d\phi ds \rho g y \sin \theta$$  (3)

where $x d\phi ds$ is the incremental surface area $dA$ on which the pressure at level $y$ is acting, and $\sin \theta$ gives the upward vertical force component due to that action. The substitutions $ds = dx/\sin \theta$, $y = x \tan \theta$, and $\tan \theta = R/h$ allow Eq. (3) to be put in the readily integrable form

$$F_\uparrow = \rho gh \int_0^{2\pi} \int_0^R \frac{x^2}{R} dx \ d\phi$$

which gives

$$F_\uparrow = \frac{1}{3} \pi R^2 \rho g h$$  (4)

Equation (4) represents the net upward hydrostatic force acting on the inner sloping surface of the cone. This upward force is twice the weight of fluid, and when subtracted from the downward triple-weight force acting on the base the result is the weight of fluid; that is, Eq. (4) subtracted from Eq. (2) gives Eq. (1), or

$$F_\downarrow - F_\uparrow = \frac{1}{3} \pi R^2 \rho g$$

This result puts the hydrostatic paradox in its proper perspective—while it is true that the force due to fluid pressure on the bottom of a vessel can be markedly different from the weight of fluid in the vessel, it is only true because the fluid pressure forces acting on the inner sloping sides have not been taken into account. When all of the vertical forces due to the varying fluid pressures are considered, the net result must be the weight of fluid, and the hydrostatic paradox becomes the interesting but incomplete observation that it is.

Reference

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**An Interesting Qualitative Problem**

Warning—This presentation contains a graphic discussion of thermodynamic phenomena that were observed in the drinking of hot tea, and may not be suitable for students whose introductory physics course omitted the study of heat transfer. Professorial discretion advised.

My wife and I are sitting down for dinner at a Chinese restaurant and the waitress brings us a pot of tea. The water for the tea was probably boiling when it was poured into the pot. The teapot is made of sheet metal that is a millimeter or less thick and only the handle of the pot is thermally insulated. We pour hot tea into two small ceramic cups whose walls are several millimeters thick. The tea in the cups is initially too hot to drink and it takes a few minutes for the tea in the cups to cool sufficiently so that we can drink it. After drinking the tea we refill our cups by pouring tea from the teapot. Again, the freshly poured tea is initially too hot to drink and we must wait a few minutes for it to cool. We observe the same thing when we pour our third cups of tea.

The teapot is made of thin metal (good conductor of heat) while the teacup is made of ceramic that is several times as thick as the metal and which one thinks of as being a poor conductor of heat. This would lead one to expect that tea in the teacup would cool less rapidly than tea in the teapot. Yet we observe just the opposite. Explain these observations.

**Albert A. Bartlett**, University of Colorado, Boulder, CO 80309-0390