Interaction of two equal vortices on a $\beta$ plane

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The interaction of two equal vortices under the influence of a gradient of background vorticity ($\beta$) is studied numerically and experimentally. If the initial shape and vorticity distribution of the vortices is fixed, two parameters determine the evolution: the normalized intercentroid distance $d_u = d/R$, where $R$ is the radius of the vortex; and the normalized gradient of background vorticity $\beta_u = \beta R/\omega$, where $\omega$ is the peak vorticity of the vortex. Alternate ways of identifying regimes of behavior in the parameter plane ($d_u, \beta_u$) are presented. These are applied to numerical simulations of interaction of vortices with steplike, steep and smooth vorticity profiles. It is found that the critical distance for merger decreases with increasing $\beta_u$ for all vortex types, and that vortices with smooth vorticity profile are the most merger-prone vortices. Laboratory experiments were done in a rotating water tank with a flat sloping bottom providing the $\beta$ effect. The vortices produced have a smooth vorticity profile and show the same behavior observed in the simulations, except that, as a result of viscous effects, the critical merger distance is shifted towards larger values of $d_u$.

I. INTRODUCTION

The interaction of two like-signed vortices in a two-dimensional fluid, which is otherwise at rest, has been the subject of intense research for the last three decades.

The merger of two equal vortices has been thoroughly studied. Early numerical studies on vortices with steplike vorticity distribution (i.e., uniform vorticity inside the vortex and null outside) determined that the critical distance for merger is $d/R \approx 3.3$, where $d$ is the intercentroid distance and $R$ is the vortex radius. The existence of this critical distance was later explained with a simplified analytic model, although the value obtained with the model is somewhat smaller. Direct numerical computations have shown that the efficiency of merger (that is to say, the fraction of the total circulation of the original vortices that ends up within the new vortex) varies between 80% and 90%, with a minimum at $d/R \approx 2.9$. It has also been shown that the presence of viscosity, however small, induces merger for all initial conditions.

Less idealized conditions have also been considered. One important effect is the asymmetry of the interacting vortices, either in the size or the amplitude of the vorticity. The variety of possible outcomes of the interaction is larger: In addition to the co-rotation and merger regimes observed in the symmetric interaction, new regimes are possible. These are the straining out of one of the vortices, either partial or total; and the exchange of mass without merger. Recent results suggest, however, that some of these regimes exist only when the vorticity distribution is steplike but not when it is smooth, as it is usually encountered in real flows.

A numerical study on the effect of the vorticity distribution on the interaction of two equal vortices showed that the critical distance for merger is not significantly modified, as long as the vortices have a finite region of nonzero vorticity and the vortex radius is taken as the distance where the azimuthal speed is highest. The efficiency of merger was also evaluated and it was found that the merger of vortices with smooth vorticity profiles is only slightly less efficient than that of vortices with steplike profiles.

The motivation of these studies comes partly from their relevance to large scale atmospheric and oceanic flows. These are fairly two-dimensional flows because of the geometry of the domain, the Earth’s rotation, and the density stratification. But oceanic and atmospheric flows are further influenced by the spatial variation of the Coriolis parameter produced by the Earth’s curvature, the so-called $\beta$ effect. A single vortex is affected by $\beta$ in the following way: the vortex acquires a westward drift velocity and it transfers part of its energy into Rossby waves (see, e.g., Ref. 9). The effect of $\beta$ on the interaction of two equal vortices with steplike vorticity distribution has been studied numerically in order to identify regimes of behavior. But the classification presented in Ref. 10, as well as the one for asymmetric interactions on which it is inspired, contains regime definitions that could be misleading and omits at least one regime. The likely reason for this is that the regimes of a two-dimensional flow cannot be defined as clearly as the regimes of a dynamical system of only a few degrees of freedom. As a matter of fact, we postulate that there is no unique way of classifying the regimes of interaction of finite-area vortices, and that any parameter used to classify them leads to regime boundaries that are usually vague.

In this work we study, numerically and experimentally, the effect of $\beta$ on the interaction of vortices with vorticity distributions which are not steplike. As in previous studies of vortex interaction the central objective is to identify regimes of behavior and to explain their existence. Our strategy is to analyze first a system of a few point vortices in order to find parameters which are suitable for identifying different re-
regimes of behavior and are analogous to parameters of the finite-area vortices. We will then use these parameters for the latter system, under the assumption that they can give a good indication of its state.

The rest of the paper is organized as follows. In Sec. II we discuss the motion of two and four point vortices. Then we present the numerical results for various vorticity distributions (Sec. III) and the experimental results (Sec. IV). Finally, a summary and some conclusions are given in Sec. V.

II. REGIMES OF INTERACTION FOR POINT VORTICES

In this section we review the motion of a small number of point vortices of (initially) equal circulation ($\kappa$), both with fixed and with $\beta$-modulated circulation. The advantage of this is that the regimes are well-defined and thus these may be used as a model for the identification of regimes in the case of finite-area vortices. A similar approach has been successfully used in the study of the stability of triangular and quadrupolar vortices.$^{11,12}$

A. Point vortices on the $f$ plane

Let us begin with the case of vortices of fixed, equal strength. The evolution of a group of $N$ point vortices is governed by a system of $2N$ ordinary differential equations (see, e.g., Ref. 13):

$$
\frac{dx_i}{dt} = \sum_{j \neq i}^{N} \frac{\kappa_j}{2\pi} \frac{y_i - y_j}{r_{ij}^2}, \quad \frac{dy_i}{dt} = \sum_{j \neq i}^{N} \frac{\kappa_j}{2\pi} \frac{x_i - x_j}{r_{ij}^2},
$$

where $(x_i, y_i)$ is the position of point-vortex $i$ and $r_{ij}$ is the distance between vortices $i$ and $j$. If $N=2$ the solution is trivial for any value of $\kappa$ and $r_{12}$ ($=d$): The two vortices remain separated by a constant distance while rotating at a uniform rate. Similar to this is the behavior of finite-area vortices when their separation $d$ is large compared with their radius $R$, say $d/R>4$. Since two point vortices cannot collapse into a single one, the point-vortex pair has no regime analogous to the merger of finite-area vortices.

We skip the case $N=3$ because we are interested in symmetric configurations that mimic the system of two equal finite-area vortices. The case $N=4$ has been thoroughly studied and shown to be integrable when all $\kappa_i$'s are equal.$^{14}$ Here we analyze the initial configuration shown in Fig. 1(a), so that each finite-area vortex is represented by two point vortices. One parameter determines the evolution: $d/R$ (note that here $R$ is the distance between the point vortices forming a pair and $d$ is the distance between the middle point of the two pairs). The value of the strength $\kappa (=\kappa_1 = \kappa_2)$ may hasten or slow down the evolution but has no influence on the qualitative behavior of the vortices.

The state of the system is determined by the value of the eight space coordinates of the vortices. The absolute position is, however, unimportant; therefore the six distances $l_i$ [with $i=1, \ldots, 6$, see Fig. 1(b)] uniquely determine the relative position of the vortices and thus the state of the system. Furthermore, because of the symmetry of the configuration, only three of these distances are needed. Figure 2 shows the evolution, in the parameter space $(l_1, l_2, l_3)$, of a set of initial conditions with $1<d/R<4$. Two regimes exist: (a) When $d/R>d_c$ ($d_c \approx 2.87$) the trajectories in state space are open lines which move in the region of small values of $l_1$ and larger values of $l_2$. In this case each pair is permanently formed by the same vortices: the pairs rotate around the geometric center of the system, while the vortices rotate around the corresponding pair’s middle point. This is analogous to the behavior of finite-area vortices located far apart: the vortices rotate around the middle point between them while fluid
elements in their interior rotate around the center of the vortex. (b) When \( d/R<d_c \), the trajectories in state space are closed lines which move in the region of small and large values of both \( l_1 \) and \( l_2 \). The vortices show a more collective behavior; each vortex pairs alternately with the vortex in the adjacent corner of the parallelogram. This is analogous to the behavior of finite-area vortices just above the critical distance for merger: the vortices approach each other, they exchange some mass and then move apart.

The distance \( d \) is therefore a discriminating parameter for the regimes of the four point-vortex system. Since it is analogous to the intercentroid distance of two vortex patches, this is expected to be a good indicator of the regimes exhibited by a pair of finite-area vortices.

### B. Point vortices on the \( \beta \) plane

Large-scale motions on the Earth are essentially affected by the latitudinal variation of the Coriolis parameter \( f \) (\( f = 2\Omega \sin \phi \), with \( \Omega \) the Earth’s angular speed and \( \phi \) the geographic latitude). For motions occurring on scales smaller than a few degrees of latitude the Coriolis parameter can be approximated as a constant value plus a linear variation in the latitudinal direction, i.e., \( f = f_0 + \beta \gamma \), where \( f_0 = 2\Omega \sin \phi_0 \) and \( \beta = 2\Omega \cos \phi_0/R_E \), with \( R_E \) the Earth’s radius.

The conservation of potential vorticity in an unforced, homogeneous, ideal fluid, is expressed by the following equation:

\[
\frac{D}{Dt}(\omega + \beta \gamma) = 0,
\]

where \( D/Dt = \partial / \partial t + u \partial / \partial x + v \partial / \partial y \) is the material derivative, and \( u \) and \( v \) are the velocities in east \( (x) \) and north \( (y) \) directions, respectively. This relation implies that the relative vorticity \( \omega \) of a vortex tube moving in latitudinal direction changes in order to preserve absolute vorticity.

Although a point vortex is not a solution of Eq. (2) when \( \beta \neq 0 \), insight into the \( \beta \)-plane dynamics has been gained by solving Eq. (1) with the circulations being modulated according to the principle of conservation of potential vorticity.\(^{15–17}\)

For that purpose an area is assigned to the point vortex and it is assumed that the vorticity \( \omega \) is uniform in the small patch represented by the vortex. If the “point” vortex represents an area \( A \), its circulation is then given by \( \kappa = \omega A \). The use of this and of conservation of mass \( (area) \) in Eq. (2) yields and expression for the vortex circulation as a function of space (known as “modulation”):

\[
\kappa_i = \kappa_0 - \beta A (y_1 - y_{0i}).
\]

Here \( y_{0i} \) represents the initial latitude, at which the vortex has strength \( \kappa_0 \).

For a system of just two point vortices the distance \( d \) between them is a constant of motion. This reduces the number of variables needed to describe the state of the system from four space coordinates to three; for instance, the position of the middle point and the orientation of the line joining the vortices (Fig. 3). Substituting the new variables

\[
X = \frac{x_1 + x_2}{2}, \quad Y = \frac{y_1 + y_2}{2}, \quad \alpha = \arctan \left( \frac{y_2 - y_1}{x_2 - x_1} \right),
\]

in Eq. (1), one obtains

\[
\frac{dX}{dt} = -\frac{\beta A}{4\pi} \sin^2 \alpha, \tag{4}
\]

\[
\frac{dY}{dt} = \frac{\beta A}{8\pi} \sin 2\alpha, \tag{5}
\]

\[
\frac{d\alpha}{dt} = \frac{\kappa_0}{\pi d^2} - \frac{\beta A}{\pi d^2} Y. \tag{6}
\]

Note that \( X \) does not appear on the right hand side of these equations. The evolution is thus governed by a system of two first-order ordinary differential equations (5)–(6). The system is moreover autonomous \((t \text{ does not appear on the right-hand side})\), therefore the motion of a point-vortex pair on the \( \beta \) plane is integrable, just like the motion of a point-vortex dipole.\(^{15–17}\)

For a relatively weak \( \beta \) effect with respect to the vortex circulation \((\beta A d/\kappa_0 \ll 1)\) the angular velocity of the pair can be approximated by \( \Omega = \Omega_0 = \kappa_0 / \pi d^2 \). Then \( \alpha = \Omega t \). From Eq. (5) it follows that the middle point oscillates but it has no net displacement in latitudinal direction. From Eq. (4) it follows that the middle point moves with a mean velocity \( \bar{U} = -\beta A / 8\pi \) in longitudinal direction; i.e., the pair moves westward for positive as well as for negative vortices. This result is analogous to the finite-area case, where cyclonic as well as anticyclonic monopoles have a net westward drift.

Equations (5)–(6) can be cast in Hamiltonian form:

\[
\frac{d\alpha}{dt} = \frac{\partial H}{\partial \alpha'}, \tag{7}
\]

\[
\frac{d\alpha'}{dt} = -\frac{\partial H}{\partial \alpha}, \tag{8}
\]

where
and \( \alpha' = d \alpha / dt \) is given by Eq. (6). For simplicity, it has been assumed that \( \alpha(t=0) = 0 \), which represents a pair of vortices aligned in west-east direction. This condition is also used in the numerical simulations and the laboratory experiments.

Figure 4 shows contour curves of the Hamiltonian function \( H \). The values of \( \beta, A \) and \( \omega_0 \) are such that the adimensional combination \( \beta_a = BR/\omega_0 = 0.5 \), where \( R = \sqrt{A/\pi} \).

Each curve represents the evolution of a particular initial condition along the line \( \alpha = 0 \); the initial value of \( \alpha' \) was chosen by changing the normalized separation \( d_a = d/R \) in the range 3–15. There are two regimes: (a) When \( \beta < 2 \omega_0 / d \) the pair rotates in a definite sense but with nonuniform angular speed, (b) when \( \beta > 2 \omega_0 / d \) the pair alternates its rotation direction.

The angle \( \alpha \) is therefore a discriminating parameter for the regimes of the point-vortex pair on the \( \beta \) plane. Since it is analogous to the orientation angle of the intercentroid line of two vortex patches, this is expected to be a good indicator of the regimes exhibited by vortex couples.

The inclusion of the \( \beta \) modulation on the evolution equations of a system of point vortices can be viewed as a perturbation, whose amplitude depends on the value of the parameter \( \beta_a \). Consider for instance the four point-vortex system discussed before. Figure 5(a) shows continuous lines four phase-space trajectories when \( \beta = 0 \), two for each regime. The corresponding trajectories when \( \beta \neq 0 \) are shown with dotted lines. When the initial condition is well within a particular regime of the unperturbed system, the "perturbed" trajectory follows closely the unperturbed one. In contrast, when the initial condition is close to the regime boundary the perturbed trajectories differ greatly from the unperturbed ones. Similarly, the use of the four point-vortices of Fig. 1(a) may be interpreted as a (strong) perturbation of Eqs. (4)–(6) which are valid for two point-vortices on the \( \beta \) plane. Figure 5(b) shows the phase-space trajectories of the four point-vortex system (thick lines) on top of the phase trajectories of the two point-vortex system (thin lines). These two examples illustrate a well-known characteristic of integrable systems of a few degrees of freedom; namely, that under perturbation the boundaries between different regimes of behavior become porous and vague.18

III. NUMERICAL RESULTS

A. Vortex-in-cell model

Equation (2) is solved numerically using a vortex-in-cell model. The outline of the method is the following: The initial vorticity field \( \omega(x,y) \) is approximated by a set of point vortices distributed regularly on the whole numerical domain; therefore, initially most point vortices (95%) have zero circulation. The area \( s \) represented by each point vortex is equal [the circulations are thus \( \kappa_k = s \omega(x_k,y_k) \), where \( (x_k,y_k) \) is the point-vortex position]. The flow evolution takes place within a rectangular region covered by a Cartesian grid; the vorticity on grid points is calculated by adding the contributions of all the point-vortices within neighboring cells (each contribution being computed with a biquadratic interpola-
tion, see, e.g., Hockney and Eastwood[19]. The stream function is obtained by inverting the Poisson equation $\nabla^2 \psi = -\omega$ on the grid with the Fourier-analysis-and-cyclic-reduction method; the value of $\psi$ along the boundary is taken as constant (free-slip boundary conditions). The velocity field is evaluated from the stream function using second-order centered differences. Then the velocity of each point vortex is determined using the same biquadratic interpolation and the positions of the point vortices are advanced in time with a second order Runge–Kutta scheme. Finally, the circulation of the point vortices is updated after every time-step according to the modulation equation (3) in order to preserve potential vorticity.

In all the simulations presented here we used a square grid of $256 \times 256$ points, and there were 10 grid points per radius of the circle of nonvanishing initial relative vorticity. We chose this resolution in order to have the boundaries far away from the vortices and minimize the effect of the free-slip condition (streamline deformation in the interior). The low number of grid points in the vortex might rise concerns about the effect of numerical diffusion and about the model’s ability to resolve the filamentation associated with the interaction processes. The former was measured by computing the evolution of a single vortex on the $f$ plane and it was found that the vorticity profile undergoes virtually no deformation in the time span used in the simulations reported here. The latter was tested by computing the interaction of unequal vortices with a resolution as low as four grid points per vortex radius; our results show the same regimes observed in contour-dynamics simulations.\(^5\)

B. The numerical experiments

In the initial condition both vortices are circular and have equal size and vorticity distribution. The latter is given by

$$\omega = \begin{cases} \omega_0 \left[ 1 - \left( \frac{r}{a} \right)^n \right] & \text{for } r < a \\ 0 & \text{for } r > a \end{cases}$$

(10)

Three vorticity profiles will be considered: (a) steplike, $n \to \infty$; (b) steep, $n = 6$; and (c) smooth, $n = 2$ (see Fig. 6). Hereafter the terms “steplike,” “steep,” and “smooth” will refer to these profiles. We also make the following definition: The radius of the vortex is the radius where the maximum velocity is reached. Therefore $R = a$ for steplike vortices, $R \approx 0.91a$ for steep vortices, and $R \approx 0.81a$ for smooth vortices.

A dimensional analysis shows that, if the initial shape and vorticity distribution of the vortices is fixed, only two parameters determine the evolution of two equal vortices on the $\beta$ plane. These parameters are the initial intercentroid distance $d$ normalized by the vortex radius $R$ ($d_\beta = d/R$), and the strength of $\beta$ with respect to the intensity of the vortices ($\beta_\beta = \beta R/\omega_0$, where $\omega_0$ is the peak relative vorticity of the initial condition). The influence of the initial orientation of the intercentroid line has been assessed by studying the evolution of a vortex pair initially oriented in north–south direction. The details of the evolution are, naturally, different. The regime of behavior, however, is not changed by the initial orientation if $\beta_\beta$ is small ($<0.05$). For larger values of $\beta_\beta$ the outcome may change if the initial condition is close to a regime boundary (by close we mean at a “distance” smaller than the “distance” between adjacent points in our grid of initial conditions).

The numerical experiments were done as follows. We computed the evolution of the two vortices for a number of initial conditions uniformly distributed in a small region of the parameter space. A similar set of simulations was done for each of the three vorticity profiles described above. The values of the parameters were in the ranges $2.9 < d_\beta < 4.2$ and $0 < \beta_\beta < 0.12$. These values of $d_\beta$ were chosen because they are around the critical distance for merger when $\beta_\beta = 0$. These values of $\beta_\beta$ were chosen because with them the dynamics is dominated by the vortices; larger values produce a rapid destruction of the vortices and the interaction is hardly observed.\(^10\) It might be relevant to note that, in the ocean, midlatitude eddies have typical radii $R \approx 100$ km and peak rotational speed $U \approx 0.1$ to 2 m/s so that $0.06 < \beta_\beta < 1.2$ (the vorticity is estimated as $U/R$); whereas in the atmosphere tropical cyclones have typical radii $R \approx 500$ km and peak rotational speed $U \approx 15$ to 35 m/s so that $0.08 < \beta_\beta < 0.2$. Therefore, only intense vortices of moderate horizontal extent, both in the ocean and in the atmosphere, have a $\beta_\beta$ in the range used here.

C. Regimes of interaction

In this section we determine the regime of behavior of the pair of finite-area vortices using a small number of parameters. This endeavor has two shortcomings. In the first place, since a continuous two-dimensional flow has an infinite number of degrees of freedom its state (or regime of behavior) cannot be determined with a finite number of parameters. The discussion at the end of Sec. II illustrates this point with examples of the evolution of four point vortices on the $\beta$ plane. The actual state-space is eight-dimensional
but the evolution is represented in spaces of three and two dimensions; some trajectories thus cannot be said to belong to a particular regime. In the second place, there is some degree of arbitrariness in the parameters used, both in the choice itself and in the measurement of the value of the parameters.

We use here two ways of identifying the regimes of interaction. First way make a visual analysis of the outcome and observe how many vortices are present at the end of the simulation and which one, if any, has undergone filamentation. The regimes are defined as follows:

(1) Merger. The vortices rotate rapidly around their middle point while approaching each other. Soon a new single vortex arises surrounded by spiral arms of vorticity. The new vortex as well as the filaments are formed by mass of the two original vortices. The largest northwestward drift of the center of mass is observed in this regime [see Fig. 7(a)].

(2) Exchange. The vortices rotate and approach each other and they form filaments which are later rolled-up around the partner. The vortices then separate and the direction of mutual rotation is usually reverted [see Fig. 7(b)].

(3) Distant interaction. The two vortices wobble around each other, the distance between them increases and they undergo changes in shape. Sometimes they form filaments but these remain in the vicinity of the vortex from which they originate. The smallest northwestward drift of the center of mass is observed in this regime [see Fig. 7(c)].

Figure 8 shows these regimes in the plane \((d_*, \beta_*)\) for the three vorticity profiles. It can be seen that the merger regime is always confined to small values of \(\beta_*\), and that the area of the parameter plane occupied by the exchange regime increases as the smoothness of the vorticity profile increases.

A second classification is obtained evaluating parameters analogous to those discussed in Sec. II; namely, the intercentroid distance \(d\) and the orientation angle \(\alpha\) (which is the angle that the line joining the two vortex centers makes with respect to the east direction). These parameters are simple indicators of the following types of behavior: (a) the vortices either approach or move apart from each other, (b) the vortices either rotate always in the same sense or they undergo a rotation reversal. Although there are four possible combinations, we have never observed a decrease of the distance being associated with a rotation reversal. There are thus only three regimes as shown in Fig. 9. In this figure the behavior of the intercentroid distance and orientation angle at each initial condition is indicated by the color of the circle halves. The upper half is dark when \(d\) decreases and white when it increases; and the lower half is dark when \(\alpha\) varies monotonically and white when it varies non-monotonically.

The three regimes are defined as follows:

(1) Approach. The vortices approach each other and they rotate faster. They usually merge as in Fig. 7(a), but sometimes they do not merge.

(2) Estrangement. The vortices separate and they rotate at a slower rate but always in the same direction [see Fig. 7(b)].

(3) Rotation reversal. The pair starts rotating in one direction, then stops and changes its rotation sense; at the same time the distance increases [see Fig. 7(c)].

Figure 10 shows the projection of the phase-space trajectories of finite-area vortices onto the phase plane \((\alpha, \alpha')\) of the modulated point-vortex model. Panel (a) shows the two regimes that occur when \(\beta\) is weak, namely vortex approach (the curves with increasing \(\alpha'\)) and vortex estrangement (decreasing \(\alpha'\)). Panel (b) shows the regime that occurs when \(\beta\) is strong, namely, rotation reversal. The value of...
that divides the strong-$\beta$ regime from either of the weak-$\beta$ regimes decreases with increasing $d_\ast$. This behavior is similar to that observed in the modulated point-vortex pair, where the boundary between the regimes of rotation and oscillation is given by $\beta_\ast = 2/d_\ast$. The actual values are, however, ten times larger than they are for finite-area vortices. A similar situation has been observed for dipolar vortices on the beta plane.\textsuperscript{17} Vortices with distributed vorticity behave as modulated point-vortices but at much smaller values of $\beta$. 

FIG. 8. Regimes of the interaction of equal vortices on the $\beta$ plane as defined by visual analysis of the outcome (see text): merger (black), exchange (gray) and distant interaction (white). Each diagram corresponds to a different vorticity profile: (a) steplike, (b) steep, (c) smooth. Nondimensional initial conditions are given by the intercentroid distance ($d_\ast$) and the gradient of the background vorticity ($\beta_\ast$).

FIG. 9. Regimes of the interaction of equal vortices on the $\beta$ plane as defined by computing the evolution of intercentroid distance and orientation (see text): Approach (gray), estrangement (gray/white), and rotation reversal (white). Each diagram corresponds to a different vorticity profile: (a) steplike, (b) steep, (c) smooth. Nondimensional initial conditions are given by the intercentroid distance ($d_\ast$) and the gradient of the background vorticity ($\beta_\ast$).
D. Regimes defined by Bertrand and Carton

A classification based on computing either the circulation or the absolute vorticity has been used in previous studies. Bertrand and Carton calculated the ratio of the final to the initial absolute vorticity (i.e., relative plus planetary vorticity) for the western and the eastern vortices ($\Gamma_1$ and $\Gamma_2$, respectively). Table I shows their classification.

A critical review of this classification yields the following conclusions. Strictly speaking the regime of “complete merger” does not exist: when two vortices merge they always leave some vorticity behind in the form of spiral arms. The misleading use of the term “complete merger” appears also in the regime definitions of asymmetric interaction on the $f$ plane (see Ref. 5), although there it is correctly stated that $\Gamma_1 > 1$ and $\Gamma_2 = 0$. The vorticity loss during a “complete merger” has been observed in both numerical and experimental studies and has been quantified to be between 5% and 25% depending on the initial conditions and the vorticity distribution.

“Elastic interactions” as defined above can occur only on the $f$ plane; for if the vortices are far from each other they deform but do not undergo filamentation. The opposite happens in the $\beta$ plane: even a single vortex forms filaments while drifting. These filaments are larger for smoother vorticity profiles, but they are also formed by steplike vortices. Obviously, when two vortices are present it is not always possible to determine whether a filament is the result of the strain generated by $\beta$ or by the partner.

Finally, a regime that we have observed is not included in Table I. That in which both vortices loose vorticity, that is $\Gamma_1 < 1$ and $\Gamma_2 < 1$. This regime is also missing in the set of regimes defined for the asymmetric interaction on the $f$ plane. It must be said, however, that this regime is more clearly observed for smooth vortices than for steplike vortices.

### IV. LABORATORY EXPERIMENTS

#### A. Laboratory simulation of a $\beta$ plane

The so-called “topographic $\beta$ effect” is used to simulate in the laboratory the variation of the Coriolis parameter due to the Earth’s shape. Let the depth of the fluid $h$ be given by $h(y) = h_0 - sy$, where $y$ is the distance along some direction and $s$ is the slope of the bottom. Using the shallow water approximation and assuming a small Rossby number ($\omega f_0 \ll 1$, where $\omega$ is the relative vorticity and $f_0$ is the local value of the Coriolis parameter) the conservation of potential vorticity is given by

$$\frac{D}{Dt} \left( \omega + \frac{sf_0 y}{h_0} \right) \approx 0. \quad (11)$$

This is equivalent to the conservation of potential vorticity in a layer of uniform depth on the $\beta$ plane, Eq. (2). The equivalent $\beta$ value is obviously $sf_0/h_0$.

#### B. Apparatus

The experiments were carried out in a square tank of horizontal dimensions $100 \times 100$ cm$^2$ and 60 cm depth.

### TABLE I. Classification of regimes by Bertrand and Carton (Ref. 10).

<table>
<thead>
<tr>
<th>Regime</th>
<th>$\Gamma_1$</th>
<th>$\Gamma_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I  Complete merger</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>II Partial merger</td>
<td>$&gt;1$</td>
<td>$&lt;1$</td>
</tr>
<tr>
<td>III Complete straining-out</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>IV Partial straining-out</td>
<td>1</td>
<td>$&lt;1$</td>
</tr>
<tr>
<td>V  Elastic interactions</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Two phenomena present in finite-area vortices, but not in the modulated point-vortex pair, contribute to the difference. In the first place, the secondary vorticity field (Rossby waves) generated by the advection of ambient fluid acts back on the original vortices, advecting them and stripping them. In the second place, as the distributed vortices radiate Rossby waves they become weaker and thus the relative strength of $\beta$ increases. Support for this explanation comes from the observation that the circulations of the finite-area vortices do not change sign, whereas the point-vortex pair reverses its rotation because the circulations change sign.

![FIG. 10. Projection of the (infinite-dimensional) phase-space trajectories of two finite-area vortices on the $\beta$ plane. The thin lines are phase-space trajectories of the point-vortex model. (a) Weak $\beta$ ($\beta_s = 0.02$). (b) Strong $\beta$ ($\beta_s = 0.1$).](image-url)
The experiment was started by filling the tank with tap water up to the desired height. Then the water was spun-up to an angular velocity of 8 rpm for a period of approximately 30 min. Once the water was in solid-body rotation, vortices were generated by the "sink method," which consists in withdrawing some water during a short time. In all experiments reported here the syphoning was stopped in this particular case $t \approx 20$ s. The markers are the experimental data, the lines are analytic profiles that fit the data in the least-squares sense. The vorticity profiles, which were chosen from the family given by Eq. (10), are step-like ($n \rightarrow \infty$, continuous line), steep ($n = 6$, broken line), and smooth ($n = 2$, dotted line). Each profile is uniquely characterized by a vortex radius and a peak velocity $(R, U)$, these values are $(2.6$ cm, $2.2$ cm s$^{-1}$), $(2.7$ cm, $1.9$ cm s$^{-1}$) and $(2.8$ cm, $1.7$ cm s$^{-1}$) for the step-like, steep and smooth vortices, respectively. Since the last two profiles give a better approximation to the experimental data, in the rest of the paper the reference values will be $2.7$ cm for the vortex radius $R$ and $1.8$ cm s$^{-1}$ for the peak velocity $U$. These values are used to compute the nondimensional $\beta_d = \beta R h_0 / \omega$. The scale of the vorticity $\omega$ is estimated by $U/R$, and $\beta$ is computed as

$$\beta = \frac{s f_0}{h_0} = \frac{4 \pi s}{h_0 T},$$

where $s$ is the bottom slope, $h_0$ is the water depth at the center of the tank and $T$ is the rotation period of the table. The nondimensional values for the laboratory experiments are therefore

$$\beta_* = \frac{4 \pi s R^2}{h_0 T U},$$

$$d_* = d/R,$$

where $U = 1.8$ cm s$^{-1}$, $R = 2.7$ cm, $T = 7.5$ s. The variable parameters are the initial distance between the syphoning tubes $d$ and the bottom slope $s$ (this also changes the water depth at the center of the tank, $h_0$, which is accordingly changed in the computation of $\beta_*$).

D. Experimental results

Figure 12(a) shows the merger of two cyclonic vortices. The vortices were generated at an initial distance of 10 cm and the inclination of the bottom topography was 4%; therefore $d_* = 3.7$ and $\beta_* = 0.014$. The first frame (time runs from top to bottom) shows the vortices in the southeast (bottom left) corner, shortly after the generation was completed; the rest of the dye is a remnant of previous experiments. The next two frames show a vigorous rotation and a moderate drift of the two vortices. At the same time mass is being exchanged through the stagnation point in the middle. The fourth frame shows that the merger process is already underway, about 70 s after the start of the experiments. Finally, the last frame shows a single, almost circular vortex drifting northward.

Figure 13(a) shows the trajectories of the vortex centers, and the time evolution of the intercentroid distance $d$ and orientation angle $\alpha$. The pair has an almost constant angular speed $\Omega = 0.2$ s$^{-1}$ before merger; afterwards the angular speed increases to $\Omega = 0.6$ s$^{-1}$. The vortices move initially.
to the northwest \([\mathbf{U}, \mathbf{V}] = (-0.5, 0.4) \, \text{cm s}^{-1}\), and the vortex formed by the merger moves in almost northward direction \([\mathbf{U}, \mathbf{V}] = (-0.02, 0.4) \, \text{cm s}^{-1}\).

Figure 12(b) shows the evolution of two cyclonic vortices which merge in a very asymmetric way. The vortices were generated at an initial distance of 9.5 cm and the inclination of the bottom topography was 10%; the nondimensional experimental parameters are thus \(d_\ast = 3.5\) and \(\beta_\ast\)
=0.036. The first frame, taken 15 s after the generation was completed, shows two fairly symmetric vortices. In spite of the short time elapsed, the vortices have almost completed one rotation; the dark vortex, which occupies the northern position in the picture, was generated in the western position. In the second and third frames the asymmetry becomes increasingly evident. The last three frames show the merger: it is clear that the western vortex dominates and occupies the center while the eastern vortex is rolled up around it.

Figure 13(b) shows the trajectories of the vortex centers, and the time evolution of \( d \) and \( \alpha \). The pair has an almost constant angular speed \( \Omega = 0.3 \text{ s}^{-1} \) before merger, then this grows to \( \Omega = 0.6 \text{ s}^{-1} \); note, however, that as the eastern vortex rolls up around the western vortex \( \Omega \) becomes increasingly difficult to define. The pair moves initially northwestward \( [(U,V)=(-0.8,0.6) \text{ cm s}^{-1}] \), but the vortex formed by the merger moves southwestward \( [(U,V)=(-0.2, -0.3) \text{ cm s}^{-1}] \). Numerical simulations show a similar behavior, but they also show that this effect is transient and the vortex soon recovers its northwestward drift.

Figure 12(c) shows the evolution of two cyclonic vortices which do not merge. The vortices were generated at an initial distance of 15 cm and the inclination of the bottom topography was again 4%; the nondimensional experimental parameters are thus \( d_\infty = 5.6 \) and \( \beta_\infty = 0.014 \). The first frame shows the vortices 10 s after the syphoning was stopped and the tubes removed. A small filament of dye connects the vortices because dye was introduced in the middle point between the tubes during the syphoning period. It is thus not an indication of vorticity or mass exchange between the vortices. As a matter of fact the filament becomes so thin that it is not visible in the next frame. An actual process of mass exchange is observed in the third and fourth frames, when the vortices approach, form cusps and exchange dye filaments. From beginning to end of these experiments (a time span of about 120 s) the vortices drifted a distance of about 2.5\( d_0 \) in the northwest direction, but they performed only half a rotation about their middle point. Clearly the \( \beta \) induced drift dominates over the mutually induced rotation.

Figure 13(c) shows the trajectories of the vortex centers, and the time evolution of \( d \) and \( \alpha \). The angular speed has an almost constant value of \( \Omega = 0.06 \text{ s}^{-1} \) during the initial stage, then it decreases to about \( \Omega = 0.02 \text{ s}^{-1} \). The pair moves always in approximately northwestward direction: initially at a speed of 0.4 cm s\(^{-1}\), then the speed increases to 0.7 cm s\(^{-1}\), and it decreases to its initial value by the end of the experiment (when the vortices reach the north wall of the tank).

We did at least two experiments at every point of the parameter plane marked in Fig. 14. The results were consistent in all realizations of the experiments at any given initial condition. A striking example is that of the critical distance for merger at a slope of 10% \( (\beta_\infty = 0.036) \). Four experiments were done with initial intercentroid distance \( d = 9.5 \text{ cm} \ (d_\infty = 3.5) \), one of them is shown in Fig. 12(b). In all of them the vortices merged asymmetrically after 1.25 rotations of the pair. Similarly, three experiments were done with \( d = 10 \text{ cm} \ (d_\infty = 3.7) \) and in all of them the pair made one full rotation and then the intercentroid distance increased rapidly by a factor 3.

Figure 14 shows the outcome of all initial conditions we used. The interactions are qualitatively similar to those of the numerical simulations. For instance the critical distance for merger \( d_c \) decreases with increasing \( \beta_\infty \), although the actual
values of \( d_v \) and the magnitude of the reduction are larger. At \( \beta_\ast = 0 \), \( d_v \) is \( \approx 4 \) for the laboratory and \( \approx 3.4 \) for the simulations, whereas at \( \beta_\ast = 0.06 \) the corresponding values are \( \approx 3.2 \) and \( \approx 3.1 \), respectively. The shift towards larger values of \( d_v \) in the experiments is a consequence of viscous effects, which induces merger for initial conditions initially outside the inviscid merger regime.\(^*\) Note, however, that the agreement between the values of \( d_v \) obtained in the laboratory and in the simulations increases with increasing \( \beta_\ast \), this suggests that a large \( \beta_\ast \) overshadows the viscous effects.

V. SUMMARY AND CONCLUSIONS

We have studied, numerically and experimentally, the interaction of vortices on the \( \beta \) plane. We discuss alternate ways of identifying regimes of behavior since there is no unique way of defining them for a two dimensional flow (this has a space-state which is infinite-dimensional). We conclude that the choice of discriminating parameter has a fundamental influence on the types of regimes that can be identified. Furthermore the boundaries between regimes identified with any parameter (or set of them) are not definite and should be taken more as guides than as well-defined frontiers. Apparently the only clear-cut boundary is that between merger and no-merger (the celebrated “critical distance” for merger of equal vortices when \( \beta = 0 \)). But as \( \beta \) increases even this regime boundary gets blurred.

In the numerical simulations the vortices have equal size and vorticity distributions. This is chosen to be steplike, steep or smooth. It is found that the critical distance for merger decreases with increasing \( \beta_\ast \) for all vortex types, i.e., the presence of \( \beta \) inhibits the merger process. The merger regime occupies a somewhat larger region of the parameter plane for vortices with a smooth vorticity profile; that is to say, vortices with smooth vorticity profile merge more easily.

The vortices produced in the laboratory have a smooth vorticity profile and show a behavior similar to that observed in the simulations, except that the critical merger distance is shifted towards larger values of \( d_v \). This is a consequence of the twofold effect of viscosity; namely, the horizontal diffusion of vorticity and the vortex squeezing due to Ekman circulation induced by the no-slip condition at the bottom of the tank. Both effects contribute to the grow of the vortex radius \( R \), thus bringing the vortex pair into the merger regime.

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