Propagation and transport properties of dipolar vortices on a $\gamma$ plane

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The dynamics and transport properties of dipolar vortices on a $\gamma$ plane (a plane where the Coriolis parameter has a quadratic variation with the latitude) are studied using the modulated point-vortex model. Similarly to the $\beta$-plane case, different regimes are found for the evolution of a single dipole, depending on the initial direction of propagation $\alpha_0$. Two steadily translating couples exist: The one rotating eastward ($\alpha_0=0$) has a stable trajectory and the one rotating westward ($\alpha_0=\pi$) is unstable. For initial angles in the range $0<\alpha_0<\pi$, the couple moves along sine-like, $8$-shaped and cycloid-like trajectories. In all solutions the dynamically relevant variables (the latitude and the direction of propagation) are periodic. The advection equations of passive particles in the dipole's velocity field can be exactly written in the form of a periodically perturbed integrable Hamiltonian system. The study of transport is performed using a "dynamical-systems theory" approach. The entrainment and detrainment of fluid as a function of $\gamma$ and $\alpha_0$ are computed exactly from some invariant curves in the Poincaré map and approximately by using the Melnikov function. The exchange of mass increases with both increasing $\gamma$ and $\alpha_0$, while the rate at which this occurs has a maximum for some $\alpha_0$ and increases with $\gamma$. A major difference in particle spreading exists between dipoles which return to their initial position after an integer number of oscillations and dipoles that do not. In the former case the Poincaré map shows broad areas of unstirred fluid coinciding with the maximum radial displacement in the dipole's meandering trajectory.

I. INTRODUCTION

Many processes in the oceans and in the atmosphere owe their existence to the variation of the Coriolis parameter ($f'$) with latitude. For mesoscale phenomena it is common to approximate $f$ as a linear function with a gradient $\beta$. Near the pole, $\beta$ vanishes and the quadratic term (with coefficient $\gamma$) becomes dominant. The leading order term in the variation of the background vorticity produced by the parabolic free surface of fluid in a rotating (laboratory) tank has the same form of the $\gamma$ effect on a rotating sphere. Although one would not expect to find parabolic sea mountains, some isolated topographic features could create a distribution of background vorticity similar to that of the $\gamma$ plane (namely, closed contours of ambient vorticity). It is well known that there is an equivalence between the $\beta$ plane dynamics in geophysical flows and the dynamics in a slab plasma, where the density gradient plays the role of $\beta$. For a cylindrically confined plasma the dynamics is analogous to that of a $\gamma$ plane, but a complete equivalence depends on the density distribution.

The dynamics of coherent vortices on the $\beta$ plane have been studied intensively over the past decades. Initially, studies were devoted to monopolar vortices and, more recently, the dipolar vortex, consisting of two closely packed regions of opposite vorticity, has been increasingly studied analytically, experimentally, and numerically. Special attention has been given to the behavior of the dipole when it propagates transversally to the lines of equal ambient vorticity. To study this problem Kono and Yamagata introduced a modulated point vortex model (see also Zabusky and McWilliams and Hobson) and found that the couple meanders periodically around its equilibrium latitude. Numerical simulations and analytical studies of continuous modons have shown the same behavior. Experiments on dipolar vortices on a "topographic" $\beta$ plane (Kloosterziel et al. and Velasco Fuentes and van Heijst, hereafter called VFvH) have confirmed the meandering motion of a dipole and the stability properties of eastward and westward traveling dipoles (ETDs and WTDs for short). During the meandering motion of the dipole there is a continuous change of the streamline pattern and therefore ambient fluid can be trapped by the passing dipole, and interior fluid can be detrained as well. Using the point-vortex model Velasco Fuentes, van Heijst, and Cremers (hereafter called VFvHC) studied the exchange of mass between the dipole and the ambient fluid as a function of the gradient of ambient vorticity and the initial direction of propagation of the couple. They obtained good agreement between the analytical-numerical results and experimental observations. The present work follows that of VFVH for the propagation of the dipole and that of VFVHC for the study of the transport between the dipole and the ambient fluid.

Much less work has been done on the $\gamma$-plane dynamics. Nof studied monopolar and dipolar vortices on the $\gamma$ plane. He found exact analytical solutions equivalent to the stationary barotropic modons on the $\beta$ plane. The modon's axis is perpendicular to the gradient of ambient vorticity, but the modon is asymmetric (its total circulation is not zero). The asymmetry depends on the location of the stationary modon. Yabuki et al. used the modulated point-vortex model to study the propagation of a single dipole in a cylindrical plasma when the dipole's axis is not perpendicular to the gradient of ambient vorticity. The qualitative behavior displayed by the dipole is the same as in the $\beta$-plane case.

This paper is organized as follows. Section II deals with...
the propagation of the dipole, which is governed by a couple of ordinary differential equations; an approximate solution is obtained and compared with numerical integrations of the complete system. In Sec. III the equations of motion of passive particles are written in the form of a periodically perturbed Hamiltonian system and relevant analytical results about transport are reviewed. Results about entrainment—detrainment are presented in Sec. IV. Section V contains a summary and some final remarks.

II. PROPAGATION OF A MODULATED POINT-VORTEX DIPOLE

Large-scale motions on a rotating sphere (the Earth) are essentially affected by the latitudinal variation of the Coriolis parameter \( f \), which is defined as \( f = 2 \Omega_E \sin \phi \) with \( \Omega_E \) the Earth’s angular speed and \( \phi \) the geographic latitude. For motions occurring on scales smaller than a few degrees of latitude the Coriolis parameter can be approximated as a constant (local) value plus a linear variation in the meridional direction, i.e., \( f = f_0 + \beta y \), where \( f_0 = 2 \Omega_E \sin \phi_0 \) and \( \beta = 2 \Omega_E \cos \phi_0 R \), with \( R \) the Earth’s radius. This approximation is known as the \( \beta \)-plane model. Close to the poles, however, \( \beta \) goes to zero and the second-order term in the Taylor expansion of \( f \) becomes more important, leading to the expression:

\[
f(\gamma) = f_0 - \gamma r',
\]
where \( \gamma = \pi L^2 \gamma \) and \( r \) is the distance to the pole. This rarely used approximation is known as the \( \gamma \)-plane model. In the context of large-scale geophysical flows the \( \gamma \) plane might seem rather limited in scope, but as mentioned in Sec. I, it may turn out to be useful in a wider range of situations, such as the “topographic” \( \gamma \) plane and a cylindrical plasma.

Conservation of potential vorticity implies that the relative vorticity \( \omega \) of a vortex tube moving in meridional direction changes as expressed by the following equation:

\[
\frac{D}{Dt} (\omega - \gamma r^2) = 0,
\]
where \( D/Dt = \partial / \partial t + u \partial / \partial x + v \partial / \partial y \) is the material derivative, \( u \) and \( v \) the velocities in the \((x)\) and \((y)\) directions, respectively, and \( r^2 = x^2 + y^2 \) is the square of the distance to the pole.

Insight into the dynamics of flows with nonuniform background vorticity has been gained by studying the evolution of a few point vortices with a “modulated” circulation. In the same spirit, and knowing that a potential vortex is not a solution of (1) for \( \gamma \neq 0 \), I studied the evolution of a dipole on the \( \gamma \) plane by using two point vortices modulated according to the principle of conservation of potential vorticity. For that purpose a certain area is assigned to the “point” vortex: Under the assumption that a point vortex represents a small (undeformable) patch of vorticity, the circulation equals the (uniform) vorticity \( \omega \) multiplied by the area of the patch. If \( \pi L^2 \gamma \) is the area associated to the point vortex, its circulation is then given by \( \kappa = \omega \pi L^2 \gamma \).

Using (1) and conservation of mass one obtains

\[
\kappa_t = \kappa_t + \pi L^2 \gamma (r_0^2 - r_1^2).
\]

For a system of just two point vortices the distance \( d \) between them is a constant of motion. The evolution of the pair is therefore completely described with the position of the middle point and the direction of propagation (see Fig. 1).

\[
\frac{dr}{dt} = -U \sin \alpha, \quad \frac{d\theta}{dt} = \frac{U}{r} \cos \alpha, \quad \frac{d\alpha}{dt} = \Omega,
\]

where the magnitude of the velocity \( U \) and the rate of change of angular direction \( \Omega \) are given by

\[
U = \frac{1}{4 \pi d} (\kappa_1 - \kappa_2), \quad \Omega = \frac{1}{2 \pi d^2} (\kappa_1 + \kappa_2) - \frac{U}{r} \cos \alpha.
\]

This is a general set of equations describing the motion of two point vortices. The second term in the expression of \( \Omega \) is used for geometrical reasons: \( \alpha \) is defined as the angle of the velocity vector to the circle \( r = \text{const} \) (Fig. 1) at the position of the dipole’s center, therefore, \( \alpha \) has a constant rate of change \( \Omega = -(1/r) \cos \alpha \) for a dipole moving in a straight line. In the modulated case both \( \Omega \) and \( U \) depend on the position of the dipole through the dependence of \( \kappa \) on the radial coordinate.
One can define \( \kappa_0 = \frac{\kappa_1 - \kappa_2}{2} \) and \( a = \frac{\kappa_1}{\kappa_0} \) so that the initial velocity of the pair is equal to the velocity of a symmetric couple with circulations \( \kappa_0 \) and \( -\kappa_0 \). According to (2) and definitions given in Fig. 1 the circulation of each point vortex is given by

\[
\kappa_1 = \frac{2}{1-a} \kappa_0 - \gamma_s [r_0^2 - r^2 + d(r \cos \alpha - r_0 \cos \alpha_0)],
\]

and

\[
\kappa_2 = \frac{2a}{1-a} \kappa_0 - \gamma_s [r_0^2 - r^2 - d(r \cos \alpha - r_0 \cos \alpha_0)].
\]

As indicated in Fig. 1, \( r \) is the radial coordinate of the dipole’s center, and the subindex 0 denotes its initial value. Substituting these expressions in the definitions of \( \Omega \) and \( U \) gives

\[
\Omega = \frac{1}{2\pi d} \left[ \kappa_0 - \gamma_s d(r \cos \alpha - r_0 \cos \alpha_0) \right],
\]

and

\[
U = \frac{1}{\pi d^2} \left( \kappa_0 + \frac{1}{1-a} \kappa_0 + \gamma_s (r^2 - r_0^2) \right) - \frac{U_0}{r} \cos \alpha.
\]

By further imposing the condition \( \alpha'(0) = 0 \), one obtains two equilibrium solutions: \( \alpha(0) = 0 \) and \( \alpha(0) = \pi \), i.e., an ETD and a WTD, respectively. This sets the initial asymmetry of the dipole

\[
a = \frac{d \cos \alpha_0 - 2r_0}{d \cos \alpha_0 + 2r_0}.
\]

As in Nof’s study of continuous modons, the steady couples are asymmetric and the asymmetry depends on the distance to the pole. In our case the initial asymmetry depends also on the initial direction of propagation: An ETD has positive net circulation, a WTD has negative net circulation and couples traveling straight poleward are symmetric.

Equations (3)–(7) describe the motion of the point-vortex dipole on the \( \gamma \) plane. Note that the azimuthal coordinate does not appear on the right-hand side of any of the evolution equations; (3b) is therefore a subsidiary equation and the evolution is entirely determined by (3a) and (3c). Equivalently, in the \( \beta \)-plane case, the east coordinate is also a subsidiary relation. This is obviously a consequence of the background vorticity being constant along the zonal coordinate \( \theta \) in the \( \gamma \) plane and \( x \) in the \( \beta \) plane.

Figure 2 shows numerical integrations (with a fourth-order Runge–Kutta scheme) of Eqs. (3)–(7). Three “regimes” can be distinguished: (a) westward cycloid-like, (b) nonpropagating 8-shaped, and (c) eastward wave-like. These regimes are also observed in the \( \beta \)-plane case (see, e.g., VFvH). Note that the maximum radial displacement is larger north of the equilibrium latitude than south of it. Similarly, the azimuthal displacement of the couple is shorter during the southern part of the trajectory than during the northern part. This is a consequence of the (local) gradient of background vorticity being stronger to the south of the equilibrium position. The same variation in circulation (asymmetry) is achieved with a smaller radial displacement, and then the couple returns to the equilibrium latitude more quickly. The asymmetry in the path becomes stronger with increasing values of \( \gamma_s \).

For the case of a small inhomogeneity \( \gamma_0 / \kappa_0 \ll 1 \) the dipole’s speed is approximately constant: \( U \approx U_0 = \kappa_0 / 2\pi d \).

And if the initial radius is relatively large \( (\Delta r / r_0) \ll 1 \), the following approximation can be made for the angular velocity:

\[
\Omega = \frac{\gamma_s}{\pi d^2} [ (r_0 + \Delta r)^2 - r_0^2 ] + \frac{U_0}{r_0} \cos \alpha_0
\]

\[- \frac{U_0}{r_0 + \Delta r} \cos \alpha = \frac{\gamma_s}{\pi d^2} (2r_0 \Delta r).
\]

These approximations make it possible to reduce (3) to a single equation for \( \alpha \):

\[
\alpha'' + \omega^2 \sin \alpha = 0,
\]

where \( \omega^2 = 2 \gamma_s r_0 U_0 / \pi d^2 \). A solution should also satisfy the initial conditions \( \alpha(0) = \alpha_0 \) and \( \alpha'(0) = 0 \), where \( \alpha_0 \) is the tilting angle. This equation is of the same form as the nonlinear simple pendulum equation and has therefore the same stability properties. This implies that an ETD \( (\alpha_0 = 0) \) corresponds to a stable equilibrium: When a small perturbation is imposed...
posed, small oscillations around the equilibrium latitude arise. On the other hand, a WTD ($\alpha_0 = \pi$) is an unstable equilibrium.

A few important results can be drawn immediately from the linearized version of this equation: For small values of $\alpha_0$, the oscillations have a constant frequency $\omega$ and the azimuthal wavelength $\lambda$ and the radial amplitude $A$ of the oscillations are given by

$$\lambda = 2\pi U_0 \omega \left(1 - \alpha_0^2 / 4\right),$$

$$A = \alpha_0 U_0 / \omega.$$  \hfill (9)

Figure 3 shows a comparison between the period $T$, amplitude $A$, and wavelength $\lambda$ for the linearized solution (of the approximate equation) and direct numerical computations of the complete set of equations. For constant $\alpha_0=0.01$ [Figs. 3(a), 3(c), and 3(e)] the agreement between the linear approximation and the complete solution is very good for the range of $\gamma$ considered in this paper. For constant $\gamma_*=0.002$ [Figs. 3(b), 3(d), and 3(f)] the agreement is only good for small values of $\alpha_0$. For the later analysis of transport in Sec. III it is useful to note that the amplitude of the meandering path grows within 5% up to a value $\alpha_0=1$.

The dipole's trajectories studied in previous paragraphs have all an oscillatory character: The orientation angle $\alpha$ varies periodically between the initial direction of propagation $\alpha_0$ and $-\alpha_0$. As in the $\beta$-plane case, depending on the parameter values and the initial conditions, the dipole can enter the libration regime, where $\alpha$ increases or decreases indefinitely. To illustrate this behavior a symmetric dipole has been located initially on the pole ($r_0=0$). The dipole moves to a region of smaller ambient vorticity and acquires net positive relative vorticity, which results in a cycloid-like path in anticlockwise sense, as can be seen in Fig. 4(a). The radial amplitude of the motion decreases with increasing $\gamma_*$, as indicated in Fig. 4(b).

III. TRANSPORT BY A MEANDERING DIPOLE

A couple of oppositely signed point-vortices with constant circulations (i.e., not modulated) either move in a straight line or rotate with constant angular velocity. In both situations a constant area of fluid is advected with the couple without change of shape. The shape and size of this region of fluid depend solely on the ratio $\kappa_1/\kappa_2$ and the distance $d$ between the point vortices. The latter is a constant of the motion with or without modulation of the vortex circulations. As described in the previous section, the meandering path of the dipole is caused by the change of the vortex circulations: $\kappa_1$ decreases while $\kappa_2$ increases (in absolute value) and vice versa. As a result the shape of the “trapped” area varies continuously, and due to conservation of mass, some fluid masses are transported in turn from the interior to the exterior and reversely. This and the following sections address two main questions: (1) How the amount of fluid entrained and detrained depends on the parameters of the problem; and (2) where a fluid particle must be located to be entrained (or detrained) during the next period of the dipole’s meandering path.

A. Advection equations

The streamfunction of the flow in a frame moving with the dipole is

$$\Psi = -(1/4\pi)\left[\kappa_1 \ln(x^2 + (y - d/2)^2) + \kappa_2 \ln(x^2 + (y + d/2)^2) + \frac{1}{4} \Omega^* (x^2 + y^2) - U_y\right],$$  \hfill (11)

where $\Omega^*$ is equal to $\Omega$ except that it does not include the correction term $(U/r)\cos \alpha$; the equations describing the trajectories of fluid particles are...
\[ \frac{dx}{dt} \frac{\partial \Psi}{\partial y} \frac{dy}{dt} = -\frac{\partial \Psi}{\partial x} \] (12)

This is a Hamiltonian system with \( \Psi \) playing the role of the Hamiltonian. If the flow is steady (\( \Psi \) is time independent) particle motions are integrable, the trajectories being simply the streamlines. Time-dependent flows, however, can produce chaotic particle trajectories, at least in some regions of the flow. Using the streamfunction given by (11), the advection equations (12) become

\[ \frac{dx}{dt} = -\frac{1}{2\pi} \left( \frac{\kappa Y_+}{I_+} + \frac{\kappa Y_-}{I_-} \right) + \Omega^* y - U, \] (13)
\[ \frac{dy}{dt} = \frac{x}{2\pi} \left( \frac{\kappa Y_+}{I_+} + \frac{\kappa Y_-}{I_-} \right) - \Omega^* x, \] (14)

where the definitions \( Y_\pm = y \pm \frac{d}{2} \) and \( I_\pm = x^2 + Y_\pm^2 \) have been used. These equations, together with the evolution equations (3)-(7) form a complete set of equations for the study of particle trajectories in the velocity field of a meandering dipole on the \( \gamma \) plane. For most of the numerical calculations and the analysis that follows, this form of the equations is suitable. However, for the perturbation calculations (Melnikov theory), it is necessary to express the equations in the form of a periodically perturbed integrable Hamiltonian system:

\[ \frac{dx}{dt} = f_1(x,y) + \gamma_* g_1[x,y,r(t;\gamma_*,\alpha_0),\alpha(t;\gamma_*,\alpha_0)], \] (15)
\[ \frac{dy}{dt} = f_2(x,y) + \gamma_* g_2[x,y,r(t;\gamma_*,\alpha_0),\alpha(t;\gamma_*,\alpha_0)]. \] (16)

The functions \( f_i \) and \( g_i \) are given by

\[ f_1 = \frac{k_0}{2\pi} \left( -\frac{2}{1-a} \frac{Y_-}{I_-} + \frac{2(1+a)y}{d^2(1-a)} \frac{1}{d} \right), \] (17)
\[ f_2 = \frac{k_0 x}{2\pi} \left( \frac{2}{1-a} \frac{1}{I_-} + \frac{1}{1-a} \frac{1}{I_+} \frac{2(1+a)}{d^2(1-a)} \right), \] (18)
\[ g_1 = \frac{1}{2\pi} \left( -\frac{\kappa Y_-}{I_-} + \frac{\kappa Y_+}{I_+} + \Omega \gamma - U \gamma \right), \] (19)
\[ g_2 = \frac{1}{2\pi} \left( \frac{\kappa Y_-}{I_-} + \frac{\kappa Y_+}{I_+} - \Omega \gamma \right), \] (20)

where

\[ \kappa_1 \gamma = r^2 - r_0^2 - d(r \cos \alpha - r_0 \cos \alpha_0), \] (21)
\[ \kappa_2 \gamma = r^2 - r_0^2 + d(r \cos \alpha - r_0 \cos \alpha_0), \] (22)
\[ \Omega \gamma = \frac{2(r^2 - r_0^2)}{d^2}, \] (23)
\[ U \gamma = -(r \cos \alpha - r_0 \cos \alpha_0). \] (24)

This representation is exact.

**B. The dynamical-systems approach to transport**

For time-periodic flows a significant simplification of the description of particle motion is achieved by using the Poincaré map—the map of the particle location \([x(t_0),y(t_0)]\) to the location one period later \([x(t_0+T),y(t_0+T)]\). Loosely speaking, this corresponds to sampling the position of a particle (relative to the dipole) every time the dipole returns to its initial configuration (e.g., \( r = r_0 \) and \( \alpha = \alpha_0 \)).

For \( \gamma_0 = 0 \) (but also for \( \alpha_0 = 0, \pi \)), the stream function is time independent and a point and its mapping lie on the same streamline. The streamline patterns of this stationary flow are illustrated in Fig. 5(a). There exist two fixed points \( p_+ \) and \( p_- \) corresponding to the front and rear stagnation points of the dipole, respectively. Both are of hyperbolic type so that there is a collection of orbits forming a line that approaches \( p_- \) as \( t \to +\infty \), called the stable manifold, and a collection of orbits that emanate from \( p_+ \) (i.e., approaches \( p_+ \) as \( t \to -\infty \)), called the unstable manifold. In the unperturbed case the unstable manifold of \( p_+ \) and the stable manifold of \( p_- \) coincide and correspond to the separatrix. There are additionally two elliptic fixed points corresponding to the positions of the point vortices. The separatrix divides the flow into three regions: the free-flow region, where particles simply move from the right to the left along the unbounded streamlines, the positive-vortex core where particles rotate anticlockwise, and the negative-vortex core where the particles rotate clockwise. The cores are trapped and travel permanently with the dipolar vortex.

For \( \gamma_0 \neq 0 \), but sufficiently small, the fixed points persist and the unstable manifold of \( p_+ \) smoothly emanates from \( p_+ \) as before, but in this case undergoes strong oscillations as it approaches \( p_- \). To construct the unstable manifold numerically, it is sufficient to compute the evolution of a small line surrounding the fixed point forward in time. The line will be

**FIG. 5.** (a) Streamlines of an unperturbed point-vortex dipole. The ratio of the vortex circulations is \( \kappa/\kappa_0 = -1.1 \) and it corresponds to an ETD rotating steadily at a distance \( r_0 = 10 \) to the pole. (b) The heteroclinic tangle in the perturbed case. The thick line is the unstable manifold (the observable structure in flow visualization) and the thin line is the stable manifold. (c) The transport mechanism in the heteroclinic tangle (see text). Region ABCD is mapped to A'B'C'D'. (d) Region abed is mapped to a'b'c'd'.
stretched in the direction of the unstable manifold. The stable manifold is constructed in a similar way, but the integration is now backwards in time.

The structure that results from the intersection of the manifolds of the two hyperbolic points is called a heteroclinic tangle [Fig. 5(b)]. The intersecting manifolds create a mechanism for transport of fluid between the interior and the exterior of the vortex dipole in the following way. Let A and C be two adjacent intersections between the stable and unstable manifolds, and B a point of the stable manifold and D a point on the unstable manifold. Note that the area ABCD in Fig. 5(c) maps to the area A'B'C'D'. This is because (i) the points A,B,C,D lie on (at least) one manifold and therefore they map to points on the same manifold; and (ii) the Poincaré map preserves orientation. If the border between the fluid trapped by the cyclonic vortex and the ambient fluid is defined as p+C along the unstable manifold of p+ and cp- along the stable manifold of p-, then the area A'B'C'D' represents the fluid that will be entrained into the cyclonic vortex in the next cycle, whereas the dotted area next A represents the fluid that will be detrained. Since the flow is incompressible, the area entrained is equal to the area detrained in every cycle.

Similarly, the tangle formed by the unstable manifold of p- and the stable manifold of p+, gives rise to transport of fluid between the cyclonic and the anticyclonic vortices [Fig. 5(d)], the border between these regions being defined as p+ c along the stable manifold and cp- along the unstable manifold. In this case the area abcd is mapped to a' b' c' d', i.e., moves from the cyclonic vortex to the anticyclonic one; and the same amount of fluid [the dotted area in Fig. 5(d)] leaves the anticyclonic vortex and enters the cyclonic one.

The exchange of mass can be evaluated directly from the discrete set of points defining the manifolds. Once a single lobe is identified the area follows from $P = \int y \, dx$ along, e.g., ABC—DCA. This method is valid for every amplitude of perturbation $\gamma^*$. 

C. Melnikov theory

Without explicitly solving the advection equations (3)–(7), it is possible to predict the behavior of the stable and the unstable manifolds using the Melnikov function. This function is, up to a known normalization factor, the first-order term in the Taylor expansion about $\gamma_* = 0$ of the distance between the stable and the unstable manifolds. The Melnikov function $M(t_0)$ is defined as

$$M(t_0) = \int_{-\infty}^{\infty} \left\{ f_1[x_u(t)] g_2[y_u(t), r(t+t_0; \gamma_*, \alpha)] \alpha(t + t_0; \gamma_*, \alpha) - f_2[x_u(t)] g_1[y_u(t), r(t + t_0; \gamma_*, \alpha)] \right\} \, dt,$$

where $x_u(t) = [x_u(t), y_u(t)]$ is the particle trajectory along the separatrix of the unperturbed dipole.

The Melnikov theorem shows that a simple zero of $M(t_0)$ implies a transverse intersection of the stable and the unstable manifolds (see, e.g., Ref. 12), while one intersection implies the existence of infinitely many intersections of the manifolds (i.e., a heteroclinic tangle). The heteroclinic tangle gives rise to horseshoe maps and forms therefore the underlying mechanism for chaotic particle motion. Therefore, the Melnikov function gives a specific criterion for the existence of chaotic particle trajectories in terms of the system parameters ($\alpha_0$ and $\gamma_*$ in our case).

One can also obtain an $O(\gamma_*)$ approximation for the area of a lobe by using the Melnikov function. The area of a lobe is given by

$$\mu - \gamma_* \int_{t_0}^{t_0+T} M(t_0) \, dt_0 + O(\gamma_*^2),$$

where $t_{01}$ and $t_{02}$ are two adjacent zeros of the Melnikov function $M(t_0)$ (i.e., they correspond to adjacent intersections of the unstable and stable manifolds).

1. Some symmetries

Note that if the particle trajectory along the separatrix is chosen in such a way that $x_u(t_0 = 0) = 0$, the following symmetries hold for the time independent components of the velocity field:

$$f_1[x_u(t)] = f_1[x_u(-t)],$$
$$f_2[x_u(t)] = -f_2[x_u(-t)].$$

Then the Melnikov function $M(t_0)$ is equal to zero for all $t_0 = t^n_0$ such that the time-periodic components of the velocity field have the same symmetry:

$$g_1[x_u(t), t^n_0 + t] = g_1[x_u(-t), t^n_0 - t],$$
$$g_2[x_u(t), t^n_0 + t] = -g_2[x_u(-t), t^n_0 - t].$$

These symmetries are satisfied if $r(t^n_0 + t) = r(t^n_0 - t)$ and $\alpha(t^n_0 + t) = \pm \alpha(t^n_0 - t)$, as can be seen in Eqs. (21)–(24). Thus $t^n_0$ must correspond to an extreme radial displacement in the motion of the dipole. In the linear approximation (see the Appendix) $t^n_0 = (n + 1/2)T/2$ for $n$ an integer and $T$ the period of the dipole’s meandering motion; $M(t_0)$ has thus an infinite number of isolated zeros, two for every period of the perturbation. Similarly, for the complete system of equations there exist an infinite number of $t^n_0$ for which $M(t^n_0) = 0$ and $dM(t_0)/dt_0 \neq 0$ for all $\gamma_* \neq 0$ and $\alpha_0 \neq 0, \pi$. The zeros correspond again to the maximum radial displacement in the dipole’s motion.

IV. NUMERICAL RESULTS

A. Transport in the oscillatory regime

The heteroclinic tangle as well as the entrainment and detrainment lobes shown in Fig. 5 are schematic drawings used to illustrate the lobe dynamics. The structure of the intersecting manifolds for any value of $\gamma_*$ and $\alpha_0$ in the parameter range used here is more complicated. The main differences are that (i) the manifolds are “longer” (measured along the unperturbed separatrix) due to the large perturbation period and that (ii) any choice of the boundaries between different regions has little resemblance with those of the unperturbed case: they either leave outside a large region of the initially trapped fluid or include a large region of am-
FIG. 6. The detrainment lobe for constant \( \alpha_0=1 \) and \( \gamma_*=0.008 \), (b) 0.014, (c) 0.02, and (d) 0.026. The detrainment lobe for constant \( \gamma_*=0.02 \) and \( \alpha_0=0.2 \), (f) 0.8, (g) 1.4, and (h) 2.0. The dashed area is the fraction of the lobe that lies within the unperturbed separatrix.

Figure 6 shows the detrainment lobe for constant \( \alpha_0=1 \) and increasing \( \gamma_* \) (a) 0.008, (b) 0.014, (c) 0.02, (d) 0.026. The lobe is thin and long for small \( \gamma_* \) and reduces in length and increases in thickness with increasing \( \gamma_* \). The area of the lobe increases with \( \gamma_* \), but it occupies rather "external" areas of the cyclonic vortex half, indicating that a large portion of the core will remain trapped by the couple. Similarly, Fig. 6 also shows the detrainment lobe for constant \( \gamma_*=0.02 \) and increasing initial orientation angles \( \alpha_0=0.2, (f) 0.8, (g) 1.4, (h) 2.0 \). The lobe increases in length and in thickness with increasing \( \alpha_0 \). Obviously the lobe area also increases. The lobe now "penetrates" closer to the positive point vortex, thus reducing the size of the positive core.

Figure 7(a) shows the amount of fluid exchanged between the cyclonic half and the ambient fluid during one oscillation of the dipole computed using the Melnikov function. The area is expressed as a fraction of the area trapped by each vortex in the unperturbed case [see, e.g., Fig. 5(a)]. The initial angle is varied from 0 to \( \pi \), and \( \gamma_* \) is varied in the range 0–0.03. The area of the lobe increases with both increasing \( \gamma_* \) and \( \alpha_0 \). The area is zero for \( \alpha_0=0 \): this initial condition corresponds to the stable equilibrium (ETD) and no change of circulation occurs in the couple, and therefore no change of the form of the separatrix. For \( \alpha_0=\pi \) the area of the lobe does not go to zero, since \( \alpha_0=\pi \) corresponds to the unstable equilibrium. The area of the lobe tends to a finite value which depends on \( \gamma_* \). For \( \gamma_*=0 \) the lobe area is also zero because then there is no variation of the circulation of the vortices.

Figure 7(b) shows the exchange rate (i.e., the amount of fluid that is exchanged per unit time) between the cyclonic half and the ambient fluid. This rate is obtained by dividing the lobe area [Fig. 7(a)] by the period of the dipole's oscillation (i.e., the perturbation period). The rate is zero for \( \gamma_*=0 \) and \( \alpha_0=0 \), where the lobe area is zero, but also for \( \alpha_0=\pi \), where the period of the oscillation goes to infinity. Therefore, for every value of \( \gamma_* \neq 0 \), the exchange rate has a maximum and this occurs at the same value of \( \alpha_0 \approx 1.9 \), within the resolution of our calculations.

FIG. 7. (a) Area \( \mu \) of the lobe detrained from the cyclonic half to the ambient fluid computed using the Melnikov function. Contour interval 0.025, \( \mu \) is zero along both axes. (b) Exchange rate \( \mu^* = \mu/\text{period} \). Contour interval 0.005.
The amount of fluid detrained in one period does not depend on the period as it does in the case of a vortex pair in an oscillating strain flow, or in the dipole with prescribed perturbation, measured by the term \( \gamma A \), where \( A \) is the amplitude of the radial displacement. The fluid area detrained in the first period varies in the range:

\[
4.88 \leq \gamma A < 5.44
\]

This can be understood in the following way: the amount of detrained fluid should be related to the difference \( S \) between the area enclosed by the unperturbed separatrix and the area enclosed by the separatrix at the position of maximal asymmetry of the dipole. If the perturbation period is of the same order or greater than the orbit period "close" to the separatrix, then most of the fluid located outside the current separatrix will be advected to the dipole’s wake. However, for a small perturbation period a significant portion of this fluid will be recaptured during the same oscillation of the dipole.

The amount of fluid detrained in one period does not depend on the period as it does in the case of a vortex pair in an oscillating strain flow, or in the dipole with prescribed perturbation. The most likely reason is that on the \( \gamma \) plane the perturbation period is, for the parameter range studied in this paper, much larger than the typical time scale of dipole propagation: the length of the dipole’s trajectory during one oscillation is at least several times the distance between the point vortices.

Here the dominant factor is the amplitude of the perturbation, measured by the term \( \gamma A \), where \( A \) is the amplitude of the radial displacement. The fluid area detrained in the first period varies in the range:

\[
4.88 \leq \gamma A < 5.44
\]

The same argument explains the almost constant ratio between the amount of fluid exchanged by the two dipole halves and the fluid exchanged between one half and the ambient fluid (approximately 1.56, according to the Melnikov function calculations). Numerical computations using the asymmetric separatrices give a ratio of 1.6.

B. Transport in the libration regime

The same calculations have been performed using somewhat different values of \( r_0 \) (but all within the oscillating regime) and the overall behavior is the same. The mass exchange slightly increases with growing \( r_0 \), specially for large values of both \( \gamma A \) and \( \alpha_0 \). The larger exchange of mass is a result of the larger local gradient of ambient vorticity 2\( r_0 \gamma \). Let us briefly discuss how transport is affected when the dipole enters the "libration" regime, using as an example a symmetric dipole initially located on the pole \( (r_0=0, \alpha_0=0) \) not defined). Except for its periodic returns to the pole, the dipole is always located in areas where the background vorticity has a smaller value than at its initial location [Fig. 4(a)], consequently the positive-vortex’s circulation varies between its initial value and some smaller value (in magnitude).
The exchange of fluid therefore occurs as follows: (i) as the dipole moves equatorwards (during the first half of the period) fluid is entrained into the cyclonic half (mainly ambient fluid but also fluid from the anticyclonic half) and fluid is detrained from the anticyclonic half; (ii) as the dipole moves back to the pole (during the second half of the period) fluid is detrained from the cyclonic half, the larger fraction being ambient fluid captured during the previous stage; and fluid is entrained into the anticyclonic vortex. The core of fluid that remains trapped by the positive vortex is larger than the core of fluid carried by the negative vortex.

This process is illustrated in Fig. 9. The solid line in Fig. 9(a) shows the entrainment lobe (“the region of fluid that will be entrained in the next period”) and the dotted line shows the shape of the initial separatrix. The same lobe is shown after one-half period in Fig. 9(b), the whole lobe is contained in the new (asymmetric) separatrix and it occupies the outermost regions of the cyclonic half. When the dipole returns to the pole only a small fraction of the “entrainment” lobe remains within the cyclonic half: most has been captured by the anticyclonic half, and a fraction has returned to the free-flow region [Fig. 9(c)].

The area of the entrainment lobe increases with $\gamma$, as shown in Fig. 10 (thick solid line). The fraction of this lobe that is entrained into the anticyclonic vortex (broken line) shows the same behavior, and the fraction that is returned to the ambient fluid remains approximately constant. A negligible amount of fluid is actually trapped by the cyclonic vortex, as can be seen in Fig. 9(c).

### C. Long time spread of particles

The previous section was devoted to the determination of the amount of fluid (lobe area) that is exchanged between different regions of the flow during a single oscillation of the meandering dipole. The evolution of particles for longer periods is explored in this section. The motivation is that, on the $\gamma$ plane, all dipole solutions return to regions of the plane they have visited before, and for some particular initial conditions the dipole returns exactly to its initial position. This has consequences for the spreading of particles. On the $\beta$ plane, for example, once a patch of fluid has been detrained it remains in the wake of the dipole without undergoing much deformation anymore (see VFvHC). On the $\gamma$ plane, on the other hand, detrained fluid can be recaptured again as the dipole returns to that region of the plane.

Three particular situations are considered: (a) a dipole with an 8-shaped trajectory and zero net zonal drift, (b) a dipole that returns exactly to its initial position after one rotation around the pole and eight oscillations around its equilibrium latitude; and (c) a dipole that returns exactly to its initial position after two rotations around the pole and seventeen oscillations around its equilibrium latitude. Particles were initially placed on a regular array within the detrainment lobe of the positive vortex (“green” particles) and the negative vortex (“yellow” particles), and their positions were sampled at times $nT$, where $n$ is an integer and $T$ is the period of the dipole’s meandering.

In case (a) one observes alternating bands of ambient and interior fluid, and the particles are spread in latitudinal and in westward direction over distances larger than the scale...
FIG. 11. Positions occupied by fluid particles after a series of iterations of the Poincaré map. The green particles were originally located within the cyclonic dipole half and the yellow ones within the negative half. (a) The 8-shaped trajectory without zonal drift (\(\alpha_0 = 2.257233\)), the positions of the particles after 5, 10, 15, and 20 periods are displayed. (b) A dipole that returns exactly to its initial position after one rotation around the pole (\(\alpha_0 = 1.021516\)), the particles' positions after 57–64 periods are displayed. (c) A dipole that returns exactly to its initial position after two rotations around the pole (\(\alpha_0 = 1.136182\)), the particles' positions after 57–64 periods are displayed. In all cases \(\chi_e = 0.01\) and the white line is the dipole's trajectory.

of the trajectory of the dipole itself [indicated by a white line in Fig. 11(a)]. There is a net westward transport of fluid in spite of the dipole's zero drift. This can be understood in the following way. As the dipole moves northward, the positive vortex (which occupies the west side of the couple) becomes weaker and detrains fluid, while the negative vortex (which occupies the east side) becomes stronger and entrains fluid. On the other hand, as the dipole moves southward, the negative vortex (on the west side of the couple) becomes weaker and detrains fluid, while the positive vortex (on the east side)
becomes stronger and entrains fluid. In both cases, the vortex located on the east entrains fluid while the vortex located on the west detrains fluid, resulting therefore in a net mass transport in westward direction.

There is a major difference in the transport of particles between cases (b) and (c). In (b), 16 broad islands of stability (larger than the dipole) arise: eight correspond to the northernmost point and the rest to the eight southernmost points of the dipole’s trajectory, indicated by a white line in Fig. 11(b). The particles are distributed forming bands. Both green and yellow particles are well spread in radial direction.

For case (c) the islands have almost completely disappeared. Both species of particles occupy a band in radial direction with amplitude comparable to the amplitude of the meandering motion (white line). The particles have a broader spreading in latitudinal direction in case (b) but they have a better azimuthal spreading for case (c).

V. CONCLUSIONS

A single dipole on the $y$ plane performs a meandering motion around circles of equal ambient vorticity when its initial direction of propagation is not parallel to these lines. The equations of motion of the point-vortex couple can be reduced to an autonomous system of two ordinary differential equations, and they are therefore integrable. In all solutions the dynamically relevant variables, namely the direction of propagation and the distance to the pole, are periodic.

There exist two trivial solutions: (a) the eastward rotating dipole corresponds to a stable equilibrium, small perturbation of the initial condition leading to meandering motion with a small amplitude; and (b) the westward rotating dipole is an unstable equilibrium, and a small perturbation of the initial conditions will lead to looping motions with large latitudinal displacements. Due to spatial variations of the gradient of ambient vorticity, the amplitude of the latitudinal displacement and the azimuthal drift are larger to the north than to the south of the equilibrium latitude.

The advection equations for the dipole’s velocity field can be exactly written in the form of a periodically perturbed integrable Hamiltonian system. Recently, several techniques for the study of transport in two-dimensional maps (lobe dynamics) are used to study entrainment and detrainment of fluid during the dipole’s meandering motion.

The amount of mass exchanged during one period of the meandering motion increases with both increasing $\gamma_s$ and $\alpha_0$. The lobe area is independent of the perturbation period, and it is approximately proportional to the product $\gamma_s A$ ($A$ is the amplitude of the latitudinal displacement). The rate at which area is detrained has a maximum for some critical value $\alpha_0 \approx 1.9$. This value is the same for the range of $\gamma_s$ considered in this work and within the resolution used to represent the parameter space.

The dipoles in the $y$ plane propagate in a limited region of the plane and periodically return (close) to areas they have occupied before. As a consequence, they are able to stir the fluid more efficiently than for example dipoles on the $\beta$ plane. A major difference in spread of particles exists between dipoles which are periodic in the azimuthal variable and dipoles which are not. In the former case the Poincaré map shows broad areas of unstrirred fluid coinciding with the maximum radial displacements of the dipole’s meandering trajectory.

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APPENDIX: MELNIKOV FUNCTION FOR THE LINEAR EQUATIONS

Using the same approximations leading to (9) and (10), the perturbation to the circulations and the angular velocity are given by

$$\kappa_1 = \kappa_2 = 2 r_0^2 \Delta r,$$

$$\Omega_1 = -\frac{4 r_0}{d^2} \Delta r,$$

where $\Delta r = -\left(\frac{U_0 \alpha_0}{\omega} \sin \omega t \right)$. And the perturbation to the linear velocity is $\mathbf{v}$. These relations lead to the following periodic components of the velocity field ($g_i$):

$$g_1 = \frac{1}{2 \pi} \left[ -2 r_0 \left( \frac{1}{I_-} + \frac{1}{I_+} \right) + y \right] \Delta r,$$

$$g_2 = \frac{1}{2 \pi} x \left[ 2 r_0 \left( \frac{1}{I_-} + \frac{1}{I_+} \right) - \frac{4 r_0}{d^2} \right] \Delta r.$$

Defining $g_i = g_i^* \Delta r$, and using the symmetries present in the equations for $f_1$, $f_2$, and $I_{\pm}$ are even and $f_2$ and $x$ are odd functions of $t$ for the choice $x_s(t=0) = 0$, the Melnikov function becomes

$$M(t_0) = -(U_0 \alpha_0 / \omega) \cos \omega t_0 \int_{-\infty}^{\infty} f_1(x_s(t)) g_1^* [x_s(t)] \sin \omega t \, dt \quad \text{(A1)}$$

The integral is nonzero for all values of $\alpha_0$, $M(t_0)$ has therefore an infinite number of simple zeros [i.e., $M(t_0) = 0$ and $\partial M(t_0)/\partial t_0 \neq 0$] in every point of the parameter space ($0 < \gamma_s < 0.03$ and $0 < \alpha_0 < \pi$).


